

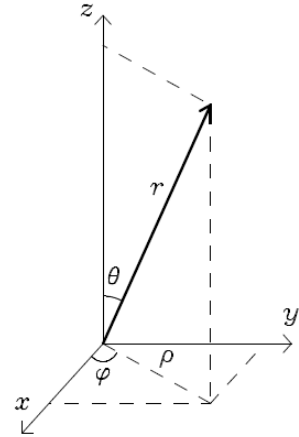
Vector Calculus Formulae

Notation

ϕ is a scalar function of a set of position coordinates. In Cartesian coordinates $\phi = \phi(x, y, z)$; in cylindrical polar coordinates $\phi = \phi(\rho, \varphi, z)$; in spherical polar coordinates $\phi = \phi(r, \theta, \varphi)$; in cases with radial symmetry $\phi = \phi(r)$. A is a vector function whose components are scalar functions of the position coordinates: in Cartesian coordinates $A = iA_x + jA_y + kA_z$, where A_x, A_y, A_z are independent functions of x, y, z .

In Cartesian coordinates ∇ ('del') $\equiv i \frac{\partial}{\partial x} + j \frac{\partial}{\partial y} + k \frac{\partial}{\partial z} \equiv \begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{bmatrix}$

$$\text{grad } \phi = \nabla \phi, \quad \text{div } A = \nabla \cdot A, \quad \text{curl } A = \nabla \times A$$



Identities

$$\text{grad}(\phi_1 + \phi_2) \equiv \text{grad } \phi_1 + \text{grad } \phi_2 \quad \text{div}(A_1 + A_2) \equiv \text{div } A_1 + \text{div } A_2$$

$$\text{grad}(\phi_1 \phi_2) \equiv \phi_1 \text{grad } \phi_2 + \phi_2 \text{grad } \phi_1$$

$$\text{curl}(A_1 + A_2) \equiv \text{curl } A_1 + \text{curl } A_2$$

$$\text{div}(\phi A) \equiv \phi \text{div } A + (\text{grad } \phi) \cdot A, \quad \text{curl}(\phi A) \equiv \phi \text{curl } A + (\text{grad } \phi) \times A$$

$$\text{div}(A_1 \times A_2) \equiv A_2 \cdot \text{curl } A_1 - A_1 \cdot \text{curl } A_2$$

$$\text{curl}(A_1 \times A_2) \equiv A_1 \text{div } A_2 - A_2 \text{div } A_1 + (A_2 \cdot \text{grad})A_1 - (A_1 \cdot \text{grad})A_2$$

$$\text{div}(\text{curl } A) \equiv 0, \quad \text{curl}(\text{grad } \phi) \equiv 0$$

$$\text{curl}(\text{curl } A) \equiv \text{grad}(\text{div } A) - \text{div}(\text{grad } A) \equiv \text{grad}(\text{div } A) - \nabla^2 A$$

$$\text{grad}(A_1 \cdot A_2) \equiv A_1 \times (\text{curl } A_2) + (A_1 \cdot \text{grad})A_2 + A_2 \times (\text{curl } A_1) + (A_2 \cdot \text{grad})A_1$$

Grad, Div, Curl and the Laplacian

	Cartesian Coordinates	Cylindrical Coordinates	Spherical Coordinates
Conversion to Cartesian Coordinates		$x = \rho \cos \varphi \quad y = \rho \sin \varphi \quad z = z$	$x = r \cos \varphi \sin \theta \quad y = r \sin \varphi \sin \theta$ $z = r \cos \theta$
Vector A	$A_x \mathbf{i} + A_y \mathbf{j} + A_z \mathbf{k}$	$A_\rho \hat{\rho} + A_\varphi \hat{\varphi} + A_z \hat{z}$	$A_r \hat{r} + A_\theta \hat{\theta} + A_\varphi \hat{\varphi}$
Gradient $\nabla \phi$	$\frac{\partial \phi}{\partial x} \mathbf{i} + \frac{\partial \phi}{\partial y} \mathbf{j} + \frac{\partial \phi}{\partial z} \mathbf{k}$	$\frac{\partial \phi}{\partial \rho} \hat{\rho} + \frac{1}{\rho} \frac{\partial \phi}{\partial \varphi} \hat{\varphi} + \frac{\partial \phi}{\partial z} \hat{z}$	$\frac{\partial \phi}{\partial r} \hat{r} + \frac{1}{r} \frac{\partial \phi}{\partial \theta} \hat{\theta} + \frac{1}{r \sin \theta} \frac{\partial \phi}{\partial \varphi} \hat{\varphi}$
Divergence $\nabla \cdot A$	$\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$	$\frac{1}{\rho} \frac{\partial(\rho A_\rho)}{\partial \rho} + \frac{1}{\rho} \frac{\partial A_\varphi}{\partial \varphi} + \frac{\partial A_z}{\partial z}$	$\frac{1}{r^2} \frac{\partial(r^2 A_r)}{\partial r} + \frac{1}{r \sin \theta} \frac{\partial A_\theta \sin \theta}{\partial \theta}$ $+ \frac{1}{r \sin \theta} \frac{\partial A_\varphi}{\partial \varphi}$
Curl $\nabla \times A$	$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix}$	$\begin{vmatrix} \frac{1}{\rho} \hat{\rho} & \hat{\varphi} & \frac{1}{\rho} \hat{z} \\ \frac{\partial}{\partial \rho} & \frac{\partial}{\partial \varphi} & \frac{\partial}{\partial z} \\ A_\rho & \rho A_\varphi & A_z \end{vmatrix}$	$\begin{vmatrix} \frac{1}{r^2 \sin \theta} \hat{r} & \frac{1}{r \sin \theta} \hat{\theta} & \frac{1}{r} \hat{\varphi} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \varphi} \\ A_r & r A_\theta & r A_\varphi \sin \theta \end{vmatrix}$
Laplacian $\nabla^2 \phi$	$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2}$	$\frac{1}{\rho} \frac{\partial}{\partial \rho} \left(\rho \frac{\partial \phi}{\partial \rho} \right) + \frac{1}{\rho^2} \frac{\partial^2 \phi}{\partial \varphi^2} + \frac{\partial^2 \phi}{\partial z^2}$	$\frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial \phi}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial \phi}{\partial \theta} \right)$ $+ \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 \phi}{\partial \varphi^2}$

Transformation of integrals

L = the distance along some curve 'C' in space and is measured from some fixed point.

S = a surface area

τ = a volume contained by a specified surface

\hat{t} = the unit tangent to C at the point P

\hat{n} = the unit outward pointing normal

A = some vector function

dL = the vector element of curve ($= \hat{t} dL$)

dS = the vector element of surface ($= \hat{n} dS$)

$$\text{Then } \int_C A \cdot \hat{t} dL = \int_C A \cdot dL$$

and when $A = \nabla\phi$

$$\int_C (\nabla\phi) \cdot dL = \int_C d\phi$$

Gauss's Theorem (Divergence Theorem)

When S defines a closed region having a volume τ

$$\int_{\tau} (\nabla \cdot A) d\tau = \int_S (A \cdot \hat{n}) dS = \int_S A \cdot dS$$

Stokes's Theorem

When C is closed and bounds the open surface S ,

$$\int_S (\nabla \times A) \cdot dS = \int_C A \cdot dL$$

also

$$\int_S (\hat{n} \times \nabla\phi) dS = \int_C \phi dL$$

Green's Theorem

$$\begin{aligned} \int_S \psi \nabla\phi \cdot dS &= \int_{\tau} \nabla \cdot (\psi \nabla\phi) d\tau \\ &= \int_{\tau} [\psi \nabla^2\phi + (\nabla\psi) \cdot (\nabla\phi)] d\tau \end{aligned}$$

Green's Second Theorem

$$\int_{\tau} (\psi \nabla^2\phi - \phi \nabla^2\psi) d\tau = \int_S [\psi(\nabla\phi) - \phi(\nabla\psi)] \cdot dS$$