Topology Homework

(1) (One point compactification) Assume that X is a non-compact connected Hausdorff space in which every point has a compact neighborhood. Define $X' := X \sqcup \{\infty\}$ as a set. You may use the fact that the intersection of a family of compact sets in a Hausdorff space is compact and the fact that the union of a finite collection of compact sets is compact.

(a) Define a topology on X' as follows: a subset $U \subset X'$ is open if (i) it is an open subset of X if $U \subset X$, and (ii) $X' - U$ is a compact subset in X if it is not a subset of X. Prove that this actually defines a topology on X'.

(b) Show that X is a subspace of X'.

(c) Show that X' is compact.

(d) Show that X' is connected.

(e) Show that if X is $\mathbb{R}^2$ with the usual topology, then X' is homeomorphic to the 2-sphere S^2.

Solution:

(a)
- $\emptyset \in \mathcal{T}$ since $\emptyset$ is an open subset of X.
  $X' \in \mathcal{T}$ since $X' - X' = \emptyset$ is a compact subset in X and X' is not a subset of X.
- Let $\{U_j\}$ satisfy condition (i), and $\{V_k\}$ satisfy condition (ii). Clearly, $U := \bigcup U_j$ still satifies condition (i), and $V := \bigcup V_k$ still satifies condition (ii) since the intersection of a family of compact sets in a Hausdorff space is compact. Note that $X' \setminus (U \cup V) = (X' \setminus U) \cap (X' \setminus V)$, where $X' \setminus U$ is closed and $X' \setminus V$ is compact in the Hausdorff space X and so the resulting set is compact in X (since it is a closed subset of a compact set). So by condition (ii), $U \cup V \subset \mathcal{T}$.
- Let $\{U_j : 1 \le j \le n\}$ satisfy condition (i), and $\{U_j : n+1 \le j \le m\}$ satisfy condition (ii). Clearly, $V := \bigcap_i^n U_j$ still satifies condition (i), and $W := \bigcap_{n+1}^m U_j$ still satifies condition (ii) since the union of finite collection of compact sets is compact. Since $X' - W$ is compact in a Hausdorff space X, it is closed, and so $W = X \setminus (X' - W)$ is open in X. Thus, we have $V \cap W \subset \mathcal{T}$.

(b) Given $U \subset \mathcal{T}$. If $U \subset X$, then $X \cap U = U$ is open in X by condition (i). If $U \not\subset X$, then $X' - U$ is compact subset of X, so $X' - U$ is closed in X and so $X \cap U = X \setminus (X' - U)$ is open in X. So, X is a subspace of X'.

(c) Let $\{\{U_i\}_{i \in I} \cup \{V_j\}_{j \in J}\}$ be any open covering of X', where each U_i satisfies condition (i) and each V_j satisfies condition (ii). Note that for each j, $V_j \setminus \{\infty\}$ is an open set in X since it is the complement of a compact (hence closed) set $X' \setminus V_j$. Also note that J is non-empty since one of the sets must cover ∞. Let V be an arbitrary element of $\{V_j\}_{j \in J}$. Then $X \setminus V$ is compact by condition (ii) and is covered by $\{U_i\}_{i \in I} \cup \{V_j \setminus \{\infty\}\}_{j \in J}$. Therefore $X \setminus V$ admits a finite subcover $\{U_i\}_{i=1}^n \cup \{V_j \setminus \{\infty\}\}_{j=1}^m$. Now clearly $\{V \cup \{U_i\}_{i=1}^n \cup \{V_j\}_{j=1}^m\}$ is a finite subcover for X'.

(d) Assume X' is not connected, i.e. $\exists$ open sets U, V such that $X' = U \cup V$ and $U \cap V = \emptyset$. WLOG, say $\infty \in U$, then V is compact in X by condition (ii), and thus V is closed. Since $V \subset X$, by condition (i), V is open in X, contradicting the fact that X is connected. Hence, X' is connected.

(e) Consider the bijective map $p : \mathbb{R} \sqcup \{\infty\} \to S^2$ defined by sending ∞ to the north pole of the sphere and the rest of the map defined by stereographic projection. By part (c), $\mathbb{R} \sqcup \{\infty\}$ is compact. S^2 is a subspace of a Hausdorff space, so it is Hausdorff. A continuous bijection from a compact space to a Hausdorff space is a homeomorphism.

(2) (a) Explain why (and how) a continuous map $f : X \to Y$ with $f(x) = y$ induces a group homomorphism $\pi_1(X, x) \to \pi_1(Y, y)$.

(b) Use the fact that $\pi_1(S^1) \cong \mathbb{Z}$ to prove Brouwer's Fixed Point Theorem: for every continuous map $f : D^2 \to D^2$, there is $a \in D^2$ such that $f(a) = a$.

Solution:

(a) If $\alpha : \mathsf{I} \to X$ is a loop at x, then $f \circ \alpha : \mathsf{I} \to Y$ is a loop at y. If $\beta : \mathsf{I} \to X$ is a loop at x such that $\beta \cong_p \alpha$, then $f \circ \beta$ is a loop at y and $f \circ \beta \cong_p f \circ \alpha$. This path-homotopy is given by $f \circ H : \mathsf{I} \times [0, 1] \to Y$ if $H : \mathsf{I} \times [0, 1] \to X$ is a path-homotopy from β to α.

(b) Suppose that there is no such fixed point, i.e. $f(x) \neq x$ for all $x \in D^2$. Then for each x, consider the half line from $f(x)$ to x. This line intersects with S^1. Let this point be denoted by $r(x)$ So define a map

$$r : B^2 \to S^1, x \mapsto r(x).$$

This map is well-defined because there is no fixed point. This map is continuous. This map r is a retraction from D^2 to S^1 and thus the inclusion $i : S^1 \to D^2$ induces an injection on fundamental groups. It is impossible since $\pi_1(S^1) = \mathbb{Z}$ and $\pi_1(D^2) = \{1\}$.

(3) Let X be any topological space, Y a Hausdorff space, and $f : X \to Y$ a continuous map. The graph of f is defined as the subspace

$$G_f := \{(x, f(x)) \in X \times Y \mid x \in X\}.$$

(a) Show that G_f is a closed subspace.
(b) Find a counter example to part (a) in the case when Y is not Hausdorff.
(c) If $f : X \to Y$ is a map and G_f is closed, then f must be continuous?

Solution:

(a) Let $(x, y) \notin G_f$, i.e $f(x) \neq y$ in Y. Since Y is Hausdorff, there are open nbhds U_y and $U_{f(x)}$ such that $U_y \cap U_{f(x)} = \varnothing$. Then consider $U := f^{-1}(U_{f(x)}) \times U_y$ which is an open nbd of (x, y). We can show that $U \cap G_f = \varnothing$ which proves that G_f is closed: let $(a, b) \in U$. Then $f(a) \in U_{f(x)}$. Since $b \in U_y$, $b \neq f(a)$.

(b) Let X be any topological space that is not Hausdorff and let $f : X \to X$ be the identity map on X. Then by HW3 question 1, we know that the diagonal $\Delta = G_f$ is not closed.

(c) I haven't gotten the complete answer yet.

(4) Let X be a topological space, and A and B compact subspaces.
(a) Show that $A \cup B$ is compact.
(b) Show that if X is Hausdorff, then $A \cap B$ is compact.
(c) Give a counterexample to part (b) in the case when X is not Hausdorff.

Solution

(a) Let $K = A \cup B$. Let $\{U_\alpha \cap K\}$ be an open covering of K where U_α's are open sets in X. Then $\{U_\alpha \cap A\}$ and $\{U_\alpha \cap B\}$ are open coverings of A and B. Since A, B are compact, we find a finite subcoverings $\{V_i \cap A, i = 1, \cdots, n\}$ and $\{W_j \cap B, j = 1, \cdots, m\}$ where $\{V_i\}, \{W_j\} \subset \{U_\alpha\}$. Then $\{V_i \cap K, W_j \cap K, i = 1, \cdots, n, j = 1, \cdots, m\}$ is a finite subcovering of K.

(b) Let $L = A \cap B$ and $K = A \cup B$. Since X is Hausdorff and A, B compact, by Thm 26.3, A and B are closed. Thus, $A \cap B$ is closed. Since $A \cap B$ is a closed subset of a compact set A, it must be compact by Thm 26.2.

Solution

(c) Let X be the Cartesian product of the real line with usual topology and the set $\{0, 1\}$ with trivial topology. Let $A = \{[a, b] \times 0\} \cup \{(a, b) \times 1\}$ and $B = \{(a, b) \times 0\} \cup \{[a, b] \times 1\}$. Note that A is compact: given an open cover of A, say $\{U_i\} = \{U_i' \times \{0, 1\}\}$, there is a finite subcover for $[a, b]$ consisting of finitely many U_i'. Similarly, B is compact. But clearly $A \cap B = (a, b) \times \{0, 1\}$ is not compact since $a \times \{0, 1\}$ is a limit point.

(5) (a) Let X be a Hausdorff space. Show that any connected subset $A \subset X$ contains one or infinitely many elements.

(b) Let A be a countable subset of $\mathbb{R}^2$. Prove that $\mathbb{R}^2 - A$ is path-connected.

Solution

(a) Say A is connected, with $1 < |A| < \infty$, say $A = \{a_1, a_2, \ldots, a_n\}$. Since X is Hausdorff, $\forall i$, $\exists U_i, V_i$ such that $a_1 \in U_i$, $a_i \in V_i$ and $U_i \cap V_i = \emptyset$. Let $U = \bigcap_i U_i$ and $V = \bigcup_i V_i$. Then U and V separate A, contradicting the assumption. Thus, A has one or infinitely many elements.

(b) Consider points $x, y \in \mathbb{R}^2 - A$. Note that there are uncountably many disjoint paths in the plane from x to y. Since A is countable, there exists a path that does not intersect A. Since x, y were arbitrary, $\mathbb{R}^2 - A$ is path-connected.

(6) Determine whether or not there is a retraction from X to A for the following spaces. If there is a retraction, describe it explicitly, using pictures if you like.

(a) X is $S^1 \times D^2$ and A is $S^1 \times S^1$.

(b) X is $S^1 \times S^1$ and $A = \{(x, x) \in X \mid x \in S^1\}$.

Solution:

(a) If there is a retraction from X to A, then $i_* : \pi_1(A) \to \pi_1(X)$ is injective. $\pi_1(X)$ is $\mathbb{Z}$ because $\pi_1(S^1 \times D^2) \cong_1 \pi_1(S^1) \times \pi_1(D^2) \cong_2 \pi_1(S^1) \times \{1\}$ ($\cong_1$ is by Thm 11.14 [L] and $\cong_2$ is by Lemma 9.2 [L]). On the other hand, $\pi_1(S^1 \times S^1)$ is $\mathbb{Z} \times \mathbb{Z}$. So it is impossible.

(b) $r : S^1 \times S^1 \to A$ defined by $(x, y) \to (x, x)$ is a retraction.

(7) Prove that a surjective map from a compact space to a Hausdorff space is a quotient map.

Solution: A surjective map from a compact space E to a Hausdorff space X is a closed map and so a quotient map: a closed set $A \subset E$ is compact since E is compact, the image of a compact subspace is compact, a compact subspace in a Hausdorff space is closed.

(8) Prove that $S^1 := \{e^{2\pi i\theta}, \theta \in \mathbb{R}\} \subset \mathbb{C}$ is homeomorphic to the quotient space obtained from $[0, 1]$ by identifying 0 and 1.

Solution: Consider the map $p : [0, 1] \to S^1, \theta \mapsto e^{2\pi i\theta}$. It factors through the quotient map $q : [0, 1] \to [0, 1]/\sim$ and there is a continuous bijection $[0, 1]/\sim \to S^1$. Since $[0, 1]$ is compact, the image of q, $[0, 1]/\sim$ is compact. S^1 is a subspace of a Hausdorff space, so it is Hausdorff. A continuous bijection from a compact space to a Hausdorff space is a homeomorphism.

References

[M] Munkres, Topology.

[S] Basic Set Theory, http://www.math.cornell.edu/~matsumura/math4530/basic set theory.pdf

[L] Lecture notes, available at http://www.math.cornell.edu/~matsumura/math4530/math4530web.html