Set Theory: Venn Diagrams

Example 1: In a group of 100 customers at Big Red's Pizza Emporium, 80 of them ordered mushrooms on their pizza and 72 of them ordered pepperoni. 60 customers ordered both mushrooms and pepperoni on their pizza.

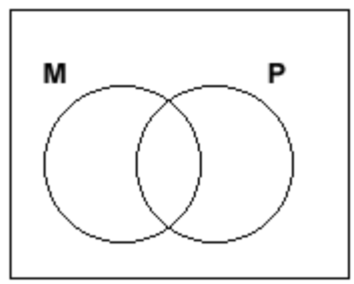
a. How many customers ordered mushrooms but no pepperoni?
b. How many customers ordered pepperoni but no mushrooms? c.

How many customers ordered neither of these two toppings?

Solution

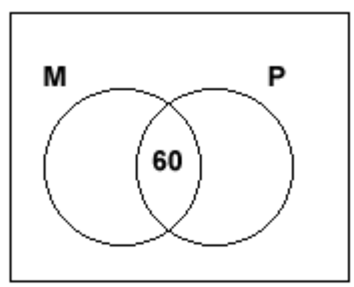
Create a Venn diagram with two sets.

To do this, first draw two intersecting circles inside a rectangle.

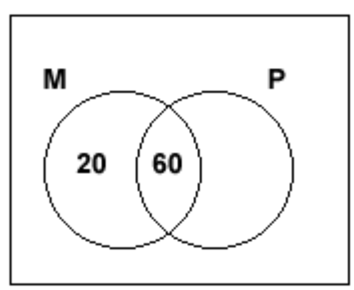


Then, make sure to work from the inside out.

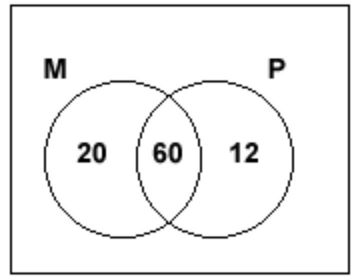
That is, first place the 60 customers that fall into the intersection of the two sets. These 60 customers ordered BOTH mushrooms AND pepperoni on their pizza, so they go into the center region.



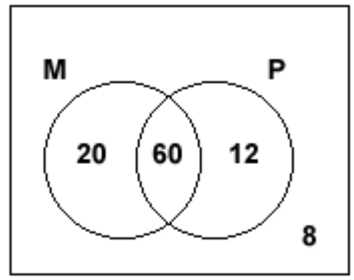
Next, we know we need to have a total of 80 customers inside the "M" circle. We already have 60 of them in there, so we have to put 20 more of them into circle for set M, being sure they are also NOT in the circle for set P.



Similarly, we know we need a total of 72 customers inside the "P" circle. We already have 60 of them in there, so we need to put 12 more into the circle for set P, being sure they are NOT in the circle for set M.



Finally, there are supposed to be 100 people on our diagram. Up to this point we have accounted for 92 of them (20 + 60 + 12), so the remaining 8 customers must go into region outside both of the circles, but still inside our rectangle.



Now, we can answer the questions.

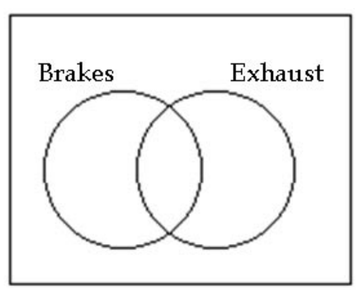
a. 20 ordered mushrooms but not pepperoni.
b. 12 ordered pepperoni but not mushrooms.
c. 8 ordered neither of these two toppings.

Example 2: At Dan's Automotive Shop, 50 cars were inspected. 23 of the cars needed new brakes, 34 needed new exhaust systems, and 6 cars needed neither repair.

a. How many cars needed both repairs?
b. How many cars needed new brakes, but not a new exhaust system?

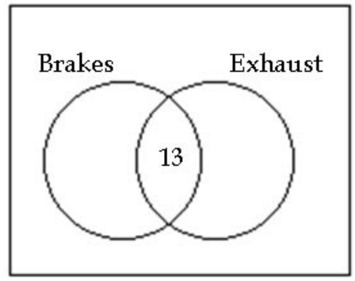
Solution

Create a Venn diagram with two sets. To do this, first draw two intersecting circles inside a rectangle. Be sure to label the circles accordingly.

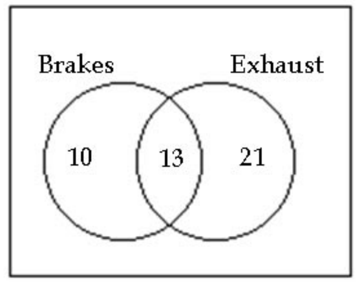


Now, work from the inside out. That is, begin by determining the number of cars in the intersection of the two sets.

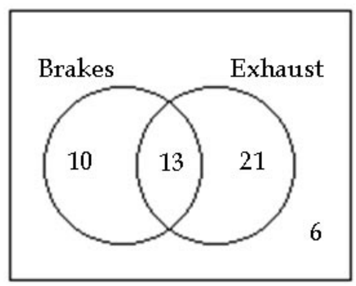
Since 6 out of the 50 cars needed no repairs, leaving 44 cars that did need repairs. 23 needed brakes and 34 needed exhaust systems. That makes 57 cars (23+34) that got worked on, which is too many; we know only 44 cars were worked on. This means is 13 cars (57-44) got counted twice, which means that 13 cars get placed into the overlapping part of the Venn diagram (the intersection). These 13 cars needed both brakes AND exhaust systems.



Then look at the circle that corresponds to Brakes. There should be 23 cars inside that circle. 13 are already accounted for, so the remaining 10 must be added into Brakes circle, but still are outside of the Exhaust circle. Likewise, 34 vehicles must appear in the Exhaust circle, so 21 more must be placed inside that circle, but not be in the Brakes circle.



Finally, 6 cars need to be indicated outside the circles, but still inside the rectangle.



Now, by looking at the completed Venn diagram, answer the original questions.

a. 13 cars needed both repairs.
b. 10 cars needed brakes, but not an exhaust system.

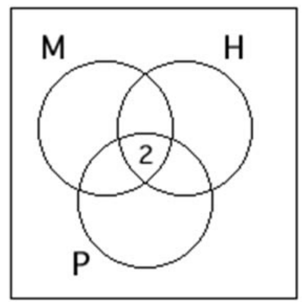
Applications with three-circle Venn diagrams are a bit longer and, consequently, a bit more involved. However, the strategy remains the same – work from the inside out.

Example 3: A survey of 85 students asked them about the subjects they liked to study. Thirty-five students liked math, 37 liked history, and 26 liked physics. Twenty liked math and history, 14 liked math and physics, and 3 liked history and physics. Two students liked all three subjects.

a. How many of these students like math or physics?
b. How many of these students didn't like any of the three subjects?
c. How many of these students liked math and history but not physics?

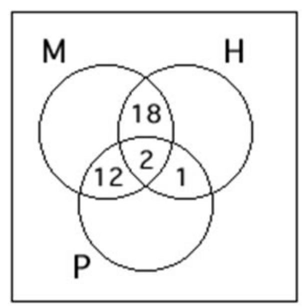
Solution

Create a Venn diagram with three sets, and label the circle M for math, H for history, and P for physics. Then make sure to work from the inside out.



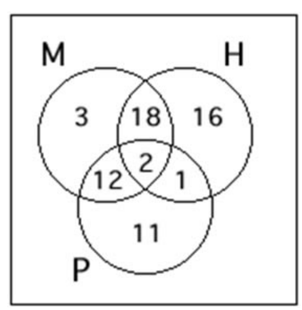
Start by placing the 2 students that like all three subjects into the center, which is the part of the diagram that represents the intersection of all three sets.

We know 20 students like math and history, so the intersection of those two sets must contain 20 students. We already have 2 of them in that intersection, so we put the remaining 18 in the intersection of the M and H circles, but not in the portion that also intersects the P circle.



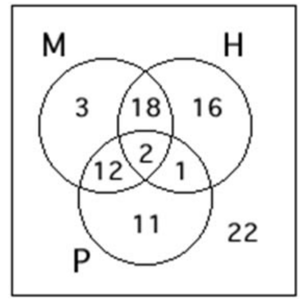
Using similar reasoning, we put 12 students and 1 student into the regions shown on the diagram.

Next, we know we need to have a total of 35 students inside the M circle. We already have 32 in there, so we put 3 students into Region I – the part of the Math circle that does not intersect with any other region.



Similarly, we put 16 students into remaining section of the H circle and 11 students into the remaining section of the P circle.

Finally, there are supposed to be 85 students included in our diagram. Up to this point we have included 63 of them, so the remaining 22 students must go portion of the diagram that is outside all of the circles, but still in the rectangle.



Now, we can answer the original questions:

a. 47 of the students like math or physics.
b. 22 of the students didn't like any of these subjects.
c. 18 of the students liked math and history but not physics.

Set Theory

Determine which of the following statements are true and which are false, and prove your answer. (NB: The symbol '\' has the same meaning as '−' in the context of set theory. Rosen uses the latter, but the former is actually more standard.)

1. If $A \subseteq B$ and $C \subseteq D$, then $A \times C \subseteq B \times D$.
2. There is a bijection between $\mathbb{R}$ and $(0, 1)$.
3. If $A \subset C$ and $A \subseteq B \subseteq C$, then either $A \subset B$ or $B \subset C$.
4. For any three sets A, B, and C, $(A \cup B) \times C = (A \times C) \cup (B \times C)$.
5. For any three sets A, B, and C, $(A \oplus B) \cup C = (A \cup C) \oplus (B \cup C)$.
6. Suppose $f, g \in A^A$, and $f \circ g = g \circ f$. Then $f \circ g = \mathrm{id}_A$.
7. If f is a one-to-one function from the set X to the set Y and $A, B \subseteq X$, then $f(A \oplus B) = f(A) \oplus f(B)$.
8. If there is a bijection from the set A to the set B and from the set C to the set D, then there is a bijection between A^C and B^D.
9. For any two sets A and B, $B \setminus (B \setminus A) = A$.
10. There exists a one-to-one function $f : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$.
11. For any four sets A, B, C, and D, $A \cup B \cap C \cup D = A \cap B \cup C \cap D$.
12. Let $f \in A^A$. Define $A_0 = A$, $A_1 = f(A)$, $A_2 = f(A_1)$, ..., $A_n = f(A_{n-1})$ for $n \geq 1$. Let $A^* = \bigcap_{j=0}^{\infty} A_j$. Then $f(A^*) \subseteq A^*$.

Set Theory Problems: Solutions

1. *True.* Suppose $(a, c) \in A \times C$. Then $a \in A$ and, since $A \subseteq B$, we have that $a \in B$. Similarly, $c \in C$ and $C \subseteq D$ implies $c \in D$. Therefore, $a \in B$ and $c \in D$, so $(a, c) \in B \times D$. We may conclude that $A \times C \subseteq B \times D$. $\square$

2. *True.* There are many such bijections; the following is just one example. Define the function $f : (0, 1) \to \mathbb{R}$ by $f(x) = \tan(\pi(x - 1/2))$. $\square$

3. *True.* Suppose not. Then $A \subset C$, but $A \not\subset B$ and $B \not\subset C$. Then it must be that $A = B$ and $B = C$, so $A = C$, contradicting the fact that A is a proper subset of C. $\square$

4. *True.* Suppose $(x, y) \in (A \cup B) \times C$. Then $x \in A \cup B$, so $x \in A$ or $x \in B$. WLOG, we may assume $x \in A$. Then, since $y \in C$, $(x, y) \in (A \times C)$, so $(x, y) \in (A \times C) \cup (B \times C)$. We may conclude that $(A \cup B) \times C \subseteq (A \times C) \cup (B \times C)$. In the other direction: Suppose $(x, y) \in (A \times C) \cup (B \times C)$. Then $(x, y) \in A \times C$ or $(x, y) \in (B \times C)$. WLOG, we may assume $(x, y) \in A \times C$. Then $x \in A$ and $y \in C$. Since $A \subseteq A \cup B$, we also have that $x \in A \cup B$. Therefore, $(x, y) \in (A \cup B) \times C$, and we may conclude that $(A \times C) \cup (B \times C) \subseteq (A \cup B) \times C$. Therefore, $(A \cup B) \times C = (A \times C) \cup (B \times C)$. $\square$

5. *False.* Let $A = \emptyset$, $B = \emptyset$, $C = \{\emptyset\}$. Then $(A \oplus B) \cup C = (\emptyset \oplus \emptyset) \cup \{\emptyset\} = \emptyset \oplus \{\emptyset\} = \{\emptyset\}$, but $(A \cup C) \oplus (B \cup C) = (\emptyset \cup \{\emptyset\}) \oplus (\emptyset \cup \{\emptyset\}) = \{\emptyset\} \oplus \{\emptyset\} = \emptyset$. $\square$

6. *False.* Let $A = \mathbb{R}$, $f(x) = x^2$ and $g(x) = x^3$. Then $f \circ g = (x^2)^3 = x^6$, and $g \circ f = (x^3)^2 = x^6$, but $f \circ g \neq \mathrm{id}_{\mathbb{R}}$.

7. *True.* Suppose that $y \in f(A \oplus B)$. Then there exists $x \in A \oplus B$ so that $f(x) = y$. Then $x \in A \setminus B$ or $x \in B \setminus A$. WLOG, we may assume $x \in A \setminus B$. Then $x \in A$, so $f(x) \in f(A)$. Suppose $f(x) \in f(B)$ as well, so that there exists a $z \in B$ with $f(x) = f(z)$. Then, since f is one-to-one, it must be that $z = x$. But then $x \in B$, contradicting the fact that $x \in A \setminus B$. Therefore, $f(A \oplus B) \subseteq f(A) \oplus f(B)$. In the other direction: Suppose $y \in f(A) \oplus f(B)$. Then $y \in f(A) \setminus f(B)$ or $y \in f(B) \setminus f(A)$. WLOG, we may assume the former. Then there is an $x \in A$ so that $f(x) = y$. Suppose $x \in B$ as well. Then $y = f(x) \in f(B)$, contradicting the fact that $y \in f(A) \setminus f(B)$. Therefore, $y \in A \setminus B$, and we may conclude that $f(A) \oplus f(B) \subseteq f(A \oplus B)$. Since we have inclusion in both directions, $f(A \oplus B) = f(A) \oplus f(B)$. $\square$

8. *True.* Suppose $f : A \to B$ and $g : C \to D$ are bijections; then g^{-1} exists. Then, for a function $h \in A^C$, we may define a function $T : A^C \to B^D$ by $T(h) = f \circ h \circ g^{-1}$. That is, for $d \in D$, $T(h)(d) = f(h(g^{-1}(d)))$. Since $g^{-1} : D \to C$, the expression $g^{-1}(d)$ makes sense; because $h : C \to A$ and $g^{-1}(d) \in C$, the expression $h(g^{-1}(d))$ makes sense; and because $h(g^{-1}(d)) \in A$ and $f : A \to B$, the expression $f(h(g^{-1}(d)))$ makes sense. It remains only to prove that $R(h) = f \circ h \circ g^{-1}$ is a bijection. To do so, we simply provide an inverse. Claim: $R : h \mapsto f^{-1} \circ h \circ g$ exists and is an inverse to T. To see this, write

$$\begin{aligned} T \circ R(h) &= f \circ (f^{-1} \circ h \circ g) \circ g^{-1} \\ &= (f \circ f^{-1}) \circ h \circ (g \circ g^{-1}) \\ &= \mathrm{id}_B \circ h \circ \mathrm{id}_D \\ &= h. \end{aligned}$$

$\square$

9. *False.* By way of counterexample, let $B = \{1, 2\}$ and $A = \{2, 3\}$. Then $B \setminus (B \setminus A) = B \setminus \{1\} = \{2\} \neq A$. $\square$

10. *True.* We could simply note that both sets are countable, and therefore equinumerous, so there exists such an injection (in fact, a bijection). However, it is more convincing to give an explicit example. Let f be defined as follows. When applied to the pair $(x, y) \in \mathbb{Z} \times \mathbb{Z}$, we first write each of $|x|$ and $|y|$ in base 8; call the resulting strings S and T. Now, if x is negative, prepend the digit '8' to S to obtain a new string S'; do the same for y and T to obtain T'. Finally, concatenate S and T with a '9' between them, and interpret the result as an integer in base 10. (Example: $f(-101_{10}, 52_{10}) = 8145964$, because $101_{10} = 1 \cdot 64_{10} + 4 \cdot 8_{10} + 5 \cdot 1_{10} = 145_8$ and $52_{10} = 6 \cdot 8_{10} + 4 \cdot 1_{10} = 64_8$.) It is easy to see that this function is one-to-one. Indeed, if $f(x, y) = z$, then z contains exactly one digit '9' when written in base 10; splitting the base 10 representation of z into the part to the left of the '9' and the part to the right of the '9' yields two nonnegative integers x' and y'; if x' begins with an '8' in base 10, then interpret the rest of it in base 8 and take its negative to obtain x; similarly, one may obtain y'.

    Here is another example of an injection $f : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$. This one is actually a bijection! First of all, define $g : \mathbb{Z} \to \mathbb{Z}^+$ by $g(x) = 2x$ if $x > 0$ and $g(x) = -2x + 1$ if $x \leq 0$. It is easy to check that this is a bijection. Defining g in this way lets us switch the problem to finding a bijection between $\mathbb{Z}^+ \times \mathbb{Z}^+$ and $\mathbb{Z}^+$. We do so by defining the "walk the antidiagonals" function described in class (and the text) – although it is modified slightly here so as always to go left-to-right instead of back-and-forth. Let $h(n, m) = (n^2 + 2nm + m^2 - n - 3m + 2)/2$. (It's not hard to obtain this formula, although it does take some thinking.) Then we can define $f(x, y) = g^{-1}(h(g(x), g(y)))$. $\square$

11. *False.* To obtain a counterexample, let $A = \{1\}$, $B = \emptyset$, $C = \{1,2\}$, and $D = \emptyset$. Then $A \cup B \cap C \cup D = \{1\} \cap \{1,2\} \cup \emptyset = \{1\} \cup \emptyset = \{1\}$, while $A \cap B \cup C \cap D = \emptyset \cup \{1,2\} \cap \emptyset = \{1,2\} \cap \emptyset = \emptyset$. □

12. *True.* Suppose $x \in A^*$. Then $x \in A_j$ for all $j \in \mathbb{N}$, so $f(x) \in A_j$ for each $j \geq 1$. Since $A_1 = f(A) \subseteq A$, we also have $f(x) \in A = A_0$. Therefore, $f(x) \in A_j$ for all $j \in \mathbb{N}$, so $f(x) \in A^*$. We may conclude that $f(A^*) \subseteq A^*$. □