Series Homework

Question 1:

Find at least 10 partial sums of the given series. Is it convergent or divergent? Explain.

Solution:

a) $\sum_{n=1}^{\infty}(0.6)^{n-1}$

$$\sum_{n=1}^{\infty}(0.6)^{n-1} = 1+(0.6)+(0.6)^2+\Lambda$$

$s_1 = 1.0,\ s_2 = 1.6,\ s_3 = 1.96,\ s_4 = 2.176,\ s_5 = 2.301,\ s_6 = 2.383,\ s_7 = 2.43,\ s_8 = 2.458,$
$s_9 = 2.475, s_{10} = 2.485$ The given series is a geometric series with $|r| = 0.6 < 1$, and it is convergent, its sum is given by

$$\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r} = \frac{1}{1-0.6} = 2.5$$

b) $\sum_{n=1}^{\infty}\frac{2n^2-1}{n^2+1}$

$$\sum_{n=1}^{\infty}\frac{2n^2-1}{n^2+1} = \frac{1}{2}+\frac{7}{5}+\frac{17}{10}+\Lambda$$

$s_1 = 0.5,\ s_2 = 1.9,\ s_3 = 3.6,\ s_4 = 5.42,\ s_5 = 7.31,\ s_6 = 9.23,\ s_7 = 11.17,\ s_8 = 13.12,$
$s_9 = 15.08, s_{10} = 17.05$

This series is a harmonic series with $\lim_{n\to\infty}\frac{2n^2-1}{n^2+1} = 2 \neq 0$. By the divergence test it is divergent series.

Question 2:

Determine whether the series is convergent or divergent. Find sum if convergent.

Solution:

a) $\sum_{n=1}^{\infty}\frac{3}{n}$. The given series is not a geometric series. Also $\lim_{n\to\infty}\frac{3}{n}=0 \Rightarrow$ divergent test fails. From example 7, page # 717 we know that $\sum_{n=1}^{\infty}\frac{1}{n}$ is a divergent series, since the sequence of partial sums $\{s_n\}$ is divergent. So $\sum_{n=1}^{\infty}\frac{3}{n}$ is also divergent.

b) $\sum_{n=1}^{\infty}\frac{3^n+2^n}{6^n}$. We have $\sum_{n=1}^{\infty}\frac{3^n+2^n}{6^n}=\sum_{n=1}^{\infty}\left(\frac{1}{2}\right)^n+\sum_{n=1}^{\infty}\left(\frac{1}{3}\right)^n$ are geometric series with $|r|<1$, thus convergent and $\sum_{n=1}^{\infty}\frac{3^n+2^n}{6^n}=\frac{1/2}{1-1/2}+\frac{1/3}{1-1/3}=\frac{3}{2}$.

c) $\sum_{n=1}^{\infty}\left(\frac{3}{n(n+3)}+\frac{5}{4^n}\right)$. We have $\sum_{n=1}^{\infty}\left(\frac{3}{n(n+3)}+\frac{5}{4^n}\right)=\sum_{n=1}^{\infty}\frac{3}{n(n+3)}+\sum_{n=1}^{\infty}\frac{5}{4^n}$.

Observe that the second series $\sum_{n=1}^{\infty}\left(\frac{5}{4^n}\right)$ is a convergent geometric series and $\sum_{n=1}^{\infty}\left(\frac{5}{4^n}\right)=\frac{5/4}{1-1/4}=\frac{5}{3}$.

We need to test the harmonic series $\sum_{n=1}^{\infty}\left(\frac{3}{n(n+3)}\right)$. We use partial fraction to get $\frac{3}{n(n+3)}=\frac{1}{n}-\frac{1}{n+3}$.

Remember the telescoping process to find $\sum_{n=1}^{\infty}\left(\frac{3}{n(n+3)}\right).=1+\frac{1}{2}+\frac{1}{3}-\frac{1}{n+3}=\frac{11}{6}$, when $n\to\infty$. Thus the harmonic series is also convergent.

The sum of the series is $\sum_{n=1}^{\infty}\left(\frac{3}{n(n+3)}+\frac{5}{4^n}\right)=\frac{11}{6}+\frac{5}{3}=\frac{7}{2}$

Question 3:

Determine using integral test whether the series convergent or divergent

Solution:

a) $\sum_{n=1}^{\infty}\frac{1}{n^4}$ is a p – series with $p = 4 > 1$, which is convergent. Now we will verify using integral test.

$\int_1^{\infty}\frac{1}{x^4}dx=\lim_{b\to\infty}\left[-\frac{1}{3x^3}\right]_1^b=\lim_{b\to\infty}\left[-\frac{1}{3b^3}+\frac{1}{3}\right]=\frac{1}{3}$ converges, thus the series converges.

b) $\sum_{n=1}^{\infty}\frac{2}{n^{0.85}}$ is a p – series with p = 0.85 < 1, which is divergent. Now we will verify using integral test.

$\int_1^{\infty}\frac{2}{x^{0.85}}dx = \lim_{b\to\infty}\left[\frac{2x^{.015}}{0.15}\right]_1^b = \lim_{b\to\infty}\left[\frac{2b^{0.15}}{0.15} - \frac{2}{0.15}\right] = \infty$ does not exist.

c) $\sum_{n=1}^{\infty}\frac{n+2}{n+1}$ is not a p – series. We will test convergence using integral test.

$\int_1^{\infty}\frac{n+2}{n+1}dx = \lim_{b\to\infty}\left[x+\ln|x+1|\right]_1^b = \infty$ does not exist. One may observe that the given series is not a decreasing series as well. The series is divergent. (One can use divergent test: $\lim_{n\to\infty}\frac{n+2}{n+1} = 1 \neq 0$)

d) $\sum_{n=1}^{\infty}\frac{\ln n}{n^2}$ is not a p – series. We will test the convergence using integral test.

$\int_2^{\infty}\frac{\ln x}{x^2}dx = \lim_{b\to\infty}\left[-\frac{\ln x}{x} - \frac{1}{x}\right]_2^b = \lim_{b\to\infty}\left[\frac{2b^{0.15}}{0.15} - \frac{2}{0.15}\right] \approx 0.85$ is finite and exists. It is convergent.

e) $\sum_{n=2}^{\infty}\frac{1}{n\ln n}$ is not a p – series. We will test the convergence using integral test.

$\int_2^{\infty}\frac{1}{x\ln x}dx = \lim_{b\to\infty}\left[\ln|\ln x|\right]_2^b = \infty$ does not exist. It is divergent.

Question 4:

Show that following series are conditionally convergent:

Solution:

a) $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{\sqrt[4]{n}}$. We have $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 1$, by ratio test the it is inconclusive. But by absolute convergence test $\lim_{n\to\infty}|a_n| = \lim_{n\to\infty}\frac{1}{n^{1/4}}$ is a *p* - series with p =1/4 < 1, divergent, on the other hand by alternating series test it is convergent, since $\lim_{n\to\infty}b_n = \lim_{n\to\infty}\frac{1}{n^{1/4}} = 0,\ b_n > b_{n+1}$. So the given series is conditionally convergent.

b) $\sum_{n=1}^{\infty}\frac{(-1)^{n-1}n}{n^2+1}$. We have $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=1$, by ratio test the it is inconclusive. But by limit comparision test (Section 11.4) $\lim_{n\to\infty}\frac{a_n}{b_n}=\lim_{n\to\infty}\frac{n/(n^2+1)}{1/n}=1>0$ is divergent since $b_n=\frac{1}{n}$ is a divergent *p* - series with p =1. On the other hand by alternating series test it is convergent, since $\lim_{n\to\infty}b_n=\lim_{n\to\infty}\frac{n}{n^2+1}=0,\ b_n>b_{n+1}$. So the given series is conditionally convergent.

Question 5:

a) Find a power series representation of $f(x)=\ln(1+x)$,what is the radius of convergence of ?

b) Find the power series representation of $f(x)=\ln(1+x^2)$

c) Find the power series representation of $f(x)=x\ln(1+x)$

Solution:

a) We have $F(x)=\frac{1}{1+x}=(1+x)^{-1}=\sum_{n=0}^{\infty}(-1)^n x^n$

Integrating on both sides w. r. to x, we find

$$\ln(1+x)=\int\frac{1}{1+x}dx=\sum_{n=0}^{\infty}(-1)^n\int x^n dx=\sum_{n=0}^{\infty}(-1)^n\frac{x^{n+1}}{n+1}+C$$

Using the initial condition *x* = 0, we find *C* = 0. Then

$$\ln(1+x)=\int\frac{1}{1+x}dx=\sum_{n=0}^{\infty}(-1)^n\frac{x^{n+1}}{n+1}=\sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^n}{n},\ with\ R=1$$

b) Using result from a) we can write $\ln(1+x^2)=\sum_{n=0}^{\infty}(-1)^n\frac{(x^2)^{n+1}}{n+1}=\sum_{n=2}^{\infty}(-1)^{n-1}\frac{x^{2n}}{n},\ with\ R=1$

c) Using result from a) we can write $x\ln(1+x)=x\sum_{n=0}^{\infty}(-1)^n\frac{(x)^{n+1}}{n+1}=\sum_{n=2}^{\infty}(-1)^n\frac{x^n}{n-1},\ with\ R=1$