Series Formulae

Arithmetic and Geometric progressions

A.P. $S_n = a + (a+d) + (a+2d) + \cdots + [a + (n-1)d] = \frac{n}{2}[2a + (n-1)d]$

G.P. $S_n = a + ar + ar^2 + \cdots + ar^{n-1} = a\frac{1-r^n}{1-r}$, $\quad \left(S_\infty = \frac{a}{1-r} \quad \text{for } |r| < 1 \right)$

(These results also hold for complex series.)

Convergence of series: the ratio test

$$S_n = u_1 + u_2 + u_3 + \cdots + u_n \quad \text{converges as} \quad n \to \infty \quad \text{if} \quad \lim_{n\to\infty} \left| \frac{u_{n+1}}{u_n} \right| < 1$$

Convergence of series: the comparison test

If each term in a series of positive terms is less than the corresponding term in a series known to be convergent, then the given series is also convergent.

Binomial expansion

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \cdots$$

If n is a positive integer the series terminates and is valid for all x: the term in x^r is ${}^nC_r x^r$ or $\binom{n}{r}$ where ${}^nC_r \equiv \frac{n!}{r!(n-r)!}$ is the number of different ways in which an unordered sample of r objects can be selected from a set of n objects without replacement. When n is not a positive integer, the series does not terminate: the infinite series is convergent for $|x| < 1$.

Taylor and Maclaurin Series

If $y(x)$ is well-behaved in the vicinity of $x = a$ then it has a Taylor series,

$$y(x) = y(a+u) = y(a) + u\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{u^2}{2!}\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + \frac{u^3}{3!}\frac{\mathrm{d}^3y}{\mathrm{d}x^3} + \cdots$$

where $u = x - a$ and the differential coefficients are evaluated at $x = a$. A Maclaurin series is a Taylor series with $a = 0$,

$$y(x) = y(0) + x\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{x^2}{2!}\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + \frac{x^3}{3!}\frac{\mathrm{d}^3y}{\mathrm{d}x^3} + \cdots$$

Power series with real variables

$$e^x = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots \qquad \text{valid for all } x$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} + \cdots + (-1)^{n+1}\frac{x^n}{n} + \cdots \qquad \text{valid for } -1 < x \le 1$$

$$\cos x = \frac{e^{ix} + e^{-ix}}{2} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \qquad \text{valid for all values of } x$$

$$\sin x = \frac{e^{ix} - e^{-ix}}{2i} = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots \qquad \text{valid for all values of } x$$

$$\tan x = x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \cdots \qquad \text{valid for } -\frac{\pi}{2} < x < \frac{\pi}{2}$$

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots \qquad \text{valid for } -1 \le x \le 1$$

$$\sin^{-1} x = x + \frac{1}{2}\frac{x^3}{3} + \frac{1.3}{2.4}\frac{x^5}{5} + \cdots \qquad \text{valid for } -1 < x < 1$$

Integer series

$$\sum_1^N n = 1 + 2 + 3 + \cdots + N = \frac{N(N+1)}{2}$$

$$\sum_1^N n^2 = 1^2 + 2^2 + 3^2 + \cdots + N^2 = \frac{N(N+1)(2N+1)}{6}$$

$$\sum_1^N n^3 = 1^3 + 2^3 + 3^3 + \cdots + N^3 = [1 + 2 + 3 + \cdots N]^2 = \frac{N^2(N+1)^2}{4}$$

$$\sum_1^\infty \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots = \ln 2 \qquad \text{[see expansion of } \ln(1+x)]$$

$$\sum_1^\infty \frac{(-1)^{n+1}}{2n-1} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots = \frac{\pi}{4} \qquad \text{[see expansion of } \tan^{-1} x]$$

$$\sum_1^\infty \frac{1}{n^2} = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots = \frac{\pi^2}{6}$$

$$\sum_1^N n(n+1)(n+2) = 1.2.3 + 2.3.4 + \cdots + N(N+1)(N+2) = \frac{N(N+1)(N+2)(N+3)}{4}$$

This last result is a special case of the more general formula,

$$\sum_1^N n(n+1)(n+2)\ldots(n+r) = \frac{N(N+1)(N+2)\ldots(N+r)(N+r+1)}{r+2}.$$

Plane wave expansion

$$\exp(ikz) = \exp(ikr\cos\theta) = \sum_{l=0}^{\infty}(2l+1)i^l j_l(kr) P_l(\cos\theta),$$

where $P_l(\cos\theta)$ are Legendre polynomials (see section 11) and $j_l(kr)$ are spherical Bessel functions, defined by

$$j_l(\rho) = \sqrt{\frac{\pi}{2\rho}} J_{l+\frac{1}{2}}(\rho), \quad \text{with } J_l(x) \text{ the Bessel function of order } l \text{ (see section 11).}$$