---

1. Determine the equilibrium temperature distribution for a one-dimensional rod with constant thermal properties with the following sources and boundary conditions:
(a) $Q = 0, \quad u(0) = 0, \quad u(L) = T.$
(f) $Q = K_0 x^2, \quad u(0) = T, \quad u'(L) = 0.$

Solution. (a) Equilibrium satisfies

$$u''(x) = 0,$$

whose general solution is

$$u = c_1 + c_2 x.$$

The boundary condition $u(0) = 0$ implies $c_1 = 0$ and $u(L) = T$ implies $c_2 = T/L$ so that

$$u = Tx/L.$$

(f) In equilibrium, u satisfies

$$u''(x) = -Q/K_0 = -x^2,$$

whose general solution (by integrating twice) is

$$u = -x^4/12 + c_1 + c_2 x.$$

The boundary condition $u(0) = T$ yields $c_1 = T$, while $u'(L) = 0$ yields $c_2 = L^3/3$. Thus

$$u = -x^4/12 + L^3 x/3 + T.$$

2. Suppose

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + x, \quad u(x,0) = f(x), \quad \frac{\partial u}{\partial x}(0,t) = \beta, \quad \frac{\partial u}{\partial x}(L,t) = 7.$$

(a) Calculate the total thermal energy in the one-dimensional rod (as a function of time).
(b) From part (a), determine a value of β for which an equilibrium exists. For this value of β, determine $\lim_{t\to\infty} u(x,t)$.

Solution. (a) Integrating the equation, we obtain:

$$\frac{d}{dt}\int_0^L u(x,t)dx = \int_0^L \left(\frac{\partial^2 u}{\partial x^2} + x\right) dx = \frac{\partial u}{\partial x}\Big|_0^L + \frac{1}{2}L^2 = 7 - \beta + \frac{1}{2}L^2.$$

Integrating in t from 0 to t, we obtain the total thermal energy

$$\int_0^L u(x,t)dx = \int_0^L f(x)dx + \left(7 - \beta + \frac{1}{2}L^2\right)t. \qquad \text{(a)}$$

(b) In order for an equilibrium to exist, $\left(7 - \beta + \frac{1}{2}L^2\right)t$ must be 0. So

$$\beta = 7 + \frac{1}{2}L^2.$$

The equilibrium satisfies

$$\phi''(x) + x = 0.$$

Its general solution (after integrating twice) is

$$\phi = -\frac{1}{6}x^3 + c_1 + c_2 x.$$

The boundary condition yields

$$c_2 = 7 + \frac{1}{2}L^2.$$

So

$$\phi = -\frac{1}{6}x^3 + c_1 + \left(7 + \frac{1}{2}L^2\right)x.$$

Since

$$\lim_{t\to\infty} u(x,t) = \phi(x),$$

using (a), we obtain

$$\begin{aligned}
\int_0^L f(x)dx &= \int_0^L u(x,t)dx \\
&= \lim_{t\to\infty} \int_0^L u(x,t)dx \\
&= \int_0^L \phi(x)dx \\
&= \int_0^L \left(-\frac{1}{6}x^3 + c_1 + \left(7 + \frac{1}{2}L^2\right)x\right)dx \\
&= -\frac{1}{24}L^4 + c_1 L + \frac{1}{2}\left(7 + \frac{1}{2}L^2\right)L^2.
\end{aligned}$$

Solving it gives

$$c_1 = \frac{\int_0^L f(x)dx - \frac{7}{2}L^2 - \frac{5}{24}L^4}{L},$$

and then

$$\phi = -\frac{1}{6}x^3 + \left(7 + \frac{1}{2}L^2\right)x + \frac{\int_0^L f(x)dx - \frac{7}{2}L^2 - \frac{5}{24}L^4}{L}.$$

3. For conduction of thermal energy, the heat flux vector is $\phi = -K_0 \nabla u$. If in addition the molecules move at an average velocity $\mathbf{V}$, a process called convection, then $\phi = -K_0\nabla u + c\rho u \mathbf{V}$. Derive the corresponding equation for heat flow, including both conduction and convection of thermal energy (assuming constant thermal properties with no sources).

Solution. Physical quantities:

- **Thermal energy density** $e(x,t)$ = the amount of thermal energy per unit volume.
- **Heat flux** $\phi(x,t)$ = the amount of thermal energy flowing across boundaries per unit surface area per unit time.

- **Heat sources** $Q(x,t)$ = heat energy per unit volume generated per unit time.
- **Temperature** $u(x,t)$.
- **Specific heat** c = the heat energy that must be supplied to a unit mass of a substance to raise its temperature one unit.
- **Mass density** $\rho(x)$ = mass per unit volume.

Conservation of heat energy:
Rate of change of heat energy in time = Heat energy flowing across boundaries per unit time + Heat energy generated insider per unit time

- heat energy = $\int_R e(x,t)dV$.
- Heat energy flowing across boundaries per unit time = $\oint \phi \cdot \mathbf{n} dS$.
- Heat energy generated insider per unit time = $\int_R Q(x,t)dV = 0$.

Then

$$\frac{\partial}{\partial t}\int_R e(x,t)dV = -\oint \phi \cdot \mathbf{n} dS.$$

The divergence theorem give

$$\frac{\partial}{\partial t}\int_R e(x,t)dV = -\int_R \nabla \cdot \phi dV.$$

and then

$$\frac{\partial e}{\partial t} = -\nabla \cdot \phi.$$

Heat energy per unit volume = $c(x)u(x,t)\rho$. So

$$e(x,t) = c(x)u(x,t)\rho.$$

It then follows that

$$c\rho\frac{\partial u}{\partial t} = -\nabla \cdot (c\rho u\mathbf{V}) + \nabla \cdot (\mathbf{K_0}\nabla \mathbf{u}).$$

and then the convection-diffusion equation

$$\frac{\partial u}{\partial t} + \nabla \cdot (u\mathbf{V}) = k\nabla^2 u,$$

where $k = \frac{K_0}{c\rho}$ is called the thermal diffusivity.

4. (d) Find the eigenvalues and the corresponding eigenfunctions of the eigenvalue problem

$$-\frac{d^2\phi}{dx^2} = \lambda\phi,$$
$$\phi(0) = 0, \quad \frac{d\phi}{dx}(L) = 0.$$

(i) If $\lambda > 0$, $\phi = c_1 \cos(\sqrt{\lambda}x) + c_2 \sin(\sqrt{\lambda}x)$. $\phi(0) = 0$ implies $c_1 = 0$, while $\frac{d\phi}{dx}(L) = 0$ implies $c_2\sqrt{\lambda}\cos(\sqrt{\lambda}L) = 0$. Thus $\sqrt{\lambda}L = -\frac{\pi}{2} + n\pi$ $(n = 1, 2, \cdots)$. Then the eigenvalues are $\lambda_n = \left(-\frac{\pi}{2} + n\pi\right)^2 / L^2$ and the corresponding eigenfunctions are $\phi_n = \sin\left(\frac{\left(-\frac{\pi}{2}+n\pi\right)x}{L}\right)$ $(n = 1, 2, \cdots)$.

(ii) If $\lambda = 0$, $\phi = c_1 + c_2 x$. $\phi(0) = 0$ implies $c_1 = 0$, while $\frac{d\phi}{dx}(L) = 0$ implies $c_2 = 0$. Thus $\lambda = 0$ is not an eigenvalue.

(ii) If $\lambda < 0$, $\phi = c_1 \exp(\sqrt{-\lambda}x) + c_2 \exp(-\sqrt{-\lambda}x)$. $\phi(0) = 0$ implies $c_1 + c_2 = 0$, while $\frac{d\phi}{dx}(L) = 0$ implies $c_1\sqrt{-\lambda}\exp(\sqrt{-\lambda}L) - c_2\sqrt{-\lambda}\exp(-\sqrt{-\lambda}L) = 0$. Solving this system for c_1, c_2 gives $c_1 = c_2 = 0$. Thus $\lambda < 0$ is not an eigenvalue.