Numerical AnalysisFormulae

Finding the zeros of equations

If the equation is $y = f(x)$ and x_n is an approximation to the root then either

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}. \qquad \text{(Newton)}$$

$$\text{or, } x_{n+1} = x_n - \frac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})} f(x_n) \qquad \text{(Linear interpolation)}$$

are, in general, better approximations.

Numerical integration of differential equations

If $\dfrac{dy}{dx} = f(x, y)$ then

$$y_{n+1} = y_n + hf(x_n, y_n) \quad \text{where } h = x_{n+1} - x_n \qquad \text{(Euler method)}$$

$$\text{Putting} \quad y^*_{n+1} = y_n + hf(x_n, y_n) \qquad \text{(improved Euler method)}$$

$$\text{then} \quad y_{n+1} = y_n + \frac{h[f(x_n, y_n) + f(x_{n+1}, y^*_{n+1})]}{2}$$

Central difference notation

If $y(x)$ is tabulated at equal intervals of x, where h is the interval, then $\delta y_{n+1/2} = y_{n+1} - y_n$ and $\delta^2 y_n = \delta y_{n+1/2} - \delta y_{n-1/2}$

Approximating to derivatives

$$\left(\frac{dy}{dx}\right)_n \approx \frac{y_{n+1} - y_n}{h} \approx \frac{y_n - y_{n-1}}{h} \approx \frac{\delta y_{n+\frac{1}{2}} + \delta y_{n-\frac{1}{2}}}{2h} \quad \text{where} \quad h = x_{n+1} - x_n$$

$$\left(\frac{d^2y}{dx^2}\right)_n \approx \frac{y_{n+1} - 2y_n + y_{n-1}}{h^2} = \frac{\delta^2 y_n}{h^2}$$

Numerical evaluation of definite integrals

Trapezoidal rule

The interval of integration is divided into n equal sub-intervals, each of width h; then

$$\int_a^b f(x)\, dx \approx h\left[c\frac{1}{2}f(a) + f(x_1) + \cdots + f(x_j) + \cdots + \frac{1}{2}f(b)\right]$$

where $h = (b - a)/n$ and $x_j = a + jh$.

Simpson's rule

The interval of integration is divided into an even number (say $2n$) of equal sub-intervals, each of width $h = (b - a)/2n$; then

$$\int_a^b f(x)\, dx \approx \frac{h}{3}[f(a) + 4f(x_1) + 2f(x_2) + 4f(x_3) + \cdots + 2f(x_{2n-2}) + 4f(x_{2n-1}) + f(b)]$$

Gauss's integration formulae

These have the general form $\int_{-1}^{1} y(x)\,\mathrm{d}x \approx \sum_{1}^{n} c_i y(x_i)$
For $n = 2$: $x_i = \pm 0{\cdot}5773$; $c_i = 1, 1$ (exact for any cubic).
For $n = 3$: $x_i = -0{\cdot}7746, 0{\cdot}0, 0{\cdot}7746$; $c_i = 0{\cdot}555, 0{\cdot}888, 0{\cdot}555$ (exact for any quintic).

Interpolation: Everett's formula

$$y(x) = y(x_0 + \theta h) \approx \overline{\theta} y_0 + \theta y_1 + \frac{1}{3!}\overline{\theta}(\overline{\theta}^2 - 1)\delta^2 y_0 + \frac{1}{3!}\theta(\theta^2 - 1)\delta^2 y_1 + \cdots$$

where θ is the fraction of the interval h $(= x_{n+1} - x_n)$ between the sampling points and $\overline{\theta} = 1 - \theta$. The first two terms represent linear interpolation.