Mathematical Logic

2.1 Propositional Formula

Exercise 2.1.
Which of the following are well formed propositional formulas?

1. $\vee pq$
2. $(\neg(p \to (q \wedge p)))$
3. $(\neg(p \to (q = p)))$
4. $(\neg(\Diamond(q \vee p)))$
5. $(p \wedge \neg q) \vee (q \to r)$
6. $p\neg r$

Solution.

Well formed formulas: 2. and 5.

*

Exercise 2.2.
Let's consider the interpretation v where $v(p) = F$, $v(q) = T$, $v(r) = T$.
Does v satisfy the following propositional formulas?

1. $(p \to \neg q) \vee \neg(r \wedge q)$
2. $(\neg p \vee \neg q) \to (p \vee \neg r)$
3. $\neg(\neg p \to \neg q) \wedge r$
4. $\neg(\neg p \to q \wedge \neg r)$

Solution.

v satisfies 1., 3. and 4.
v doesn't satisfy 2.

2.2 Truth Tables

Exercise 2.3.
Compute the truth table of $(F \vee G) \wedge \neg(F \wedge G)$.

Solution.

F	G	$F \vee G$	$F \wedge G$	$\neg(F \wedge G)$	$(F \vee G) \wedge \neg(F \wedge G)$
T	*T*	*T*	*T*	*F*	**F**
T	*F*	*T*	*F*	*T*	**T**
F	*T*	*T*	*F*	*T*	**T**
F	*F*	*F*	*F*	*T*	**F**

☞ *The formula models an exclusive or!*

*

Exercise 2.4.

Use the truth tables method to determine whether $(p \to q) \lor (p \to \neg q)$ *is valid.*

Solution.

p	q	$p \to q$	$\neg q$	$p \to \neg q$	$(p \to q) \lor (p \to \neg q)$
T	T	T	F	F	**T**
T	F	F	T	T	**T**
F	T	T	F	T	**T**
F	F	T	T	T	**T**

The formula is valid since it is satisfied by every interpretation.

*

Exercise 2.5.

Use the truth tables method to determine whether $(\neg p \lor q) \land (q \to \neg r \land \neg p) \land (p \lor r)$ *(denoted with* φ*) is satisfiable.*

Solution.

p	q	r	$\neg p \lor q$	$\neg r \land \neg p$	$q \to \neg r \land \neg p$	$(p \lor r)$	φ
T	T	T	T	F	F	T	F
T	T	F	T	F	F	T	F
T	F	T	F	F	T	T	F
T	F	F	F	F	T	T	F
F	T	T	T	F	F	T	F
F	T	F	T	T	T	F	F
F	F	T	T	F	T	T	**T**
F	F	F	T	T	T	F	F

There exists an interpretation satisfying φ*, thus* φ *is satisfiable.*

*

Exercise 2.6.
Use the truth tables method to determine whether the formula $\varphi : p \wedge \neg q \to p \wedge q$ *is a logical consequence of the formula* $\psi : \neg p$.

Solution.

p	q	$\neg p$	$p \wedge \neg q$	$p \wedge q$	$p \wedge \neg q \to p \wedge q$
T	*T*	*F*	*F*	*T*	*T*
T	*F*	*F*	*T*	*F*	*F*
F	*T*	**T**	*F*	*F*	**T**
F	*F*	**T**	*F*	*F*	**T**

$\psi \models \varphi$ *since each interpretation satisfying* psi *satisfies also* φ.

*

Exercise 2.7. ☼ ✍
Use the truth tables method to determine whether $p \to (q \wedge \neg q)$ *and* $\neg p$ *are logically equivalent.*

Solution.

p	q	$q \wedge \neg q$	$p \to (q \wedge \neg q)$	$\neg p$
T	*T*	*F*	**F**	**F**
T	*F*	*F*	**F**	**F**
F	*T*	*F*	**T**	**T**
F	*F*	*F*	**T**	**T**

The two formulas are equivalent since for every possible interpretation they evaluate to tha same truth value.

*

Exercise 2.8.
Compute the truth tables for the following propositional formulas:

- $(p \to p) \to p$
- $p \to (p \to p)$
- $p \lor q \to p \land q$
- $p \lor (q \land r) \to (p \land r) \lor q$
- $p \to (q \to p)$
- $(p \land \neg q) \lor \neg(p \leftrightarrow q)$

*

Exercise 2.9.

Use the truth table method to verify whether the following formulas are valid, satisfiable or unsatisfiable:

- $(p \to q) \land \neg q \to \neg p$
- $(p \to q) \to (p \to \neg q)$
- $(p \lor q \to r) \lor p \lor q$
- $(p \lor q) \land (p \to r \land q) \land (q \to \neg r \land p)$
- $(p \to (q \to r)) \to ((p \to q) \to (p \to r))$
- $(p \lor q) \land (\neg q \land \neg p)$
- $(\neg p \to q) \lor ((p \land \neg r) \leftrightarrow q)$
- $(p \to q) \land (p \to \neg q)$
- $(p \to (q \lor r)) \lor (r \to \neg p)$

*

Exercise 2.10.
Use the truth table method to verify whether the following logical consequences and equivalences are correct:

- $(p \to q) \models \neg p \to \neg q$
- $(p \to q) \wedge \neg q \models \neg p$
- $p \to q \wedge r \models (p \to q) \to r$
- $p \vee (\neg q \wedge r) \models q \vee \neg r \to p$
- $\neg(p \wedge q) \equiv \neg p \vee \neg q$
- $(p \vee q) \wedge (\neg p \to \neg q) \equiv q$
- $(p \wedge q) \vee r \equiv (p \to \neg q) \to r$
- $(p \vee q) \wedge (\neg p \to \neg q) \equiv p$
- $((p \to q) \to q) \to q \equiv p \to q$

2.3 Propositional Formalization

2.3.1 Formalizing Simple Sentences

Exercise 2.11.
Let's consider a propositional language where

- p *means* "Paola is happy",
- q *means* "Paola paints a picture",
- r *means* "Renzo is happy".

Formalize the following sentences:

1. "if Paola is happy and paints a picture then Renzo isn't happy"

2. "if Paola is happy, then she paints a picture"

3. "Paola is happy only if she paints a picture"

Solution.

1. $p \wedge q \rightarrow \neg r$

2. $p \rightarrow q$

3. $\neg(p \wedge \neg q)$ *..which is equivalent to* $p \rightarrow q$

☞ *The precision of formal languages avoid the ambiguities of natural languages.*

*

Exercise 2.12.
Let's consider a propositional language where

- p *means* "x is a prime number", • q *means* "x is odd".

Formalize the following sentences:

1. "x being prime is a sufficient condition for x being odd"

2. "x being odd is a necessary condition for x being prime"

Solution. *1. and 2.* $p \rightarrow q$

*

Exercise 2.13.
Let A =“Aldo is Italian” *and* B =“Bob is English”.

Formalize the following sentences:

1. “Aldo isn’t Italian”
2. “Aldo is Italian while Bob is English”
3. “If Aldo is Italian then Bob is not English”
4. “Aldo is Italian or if Aldo isn’t Italian then Bob is English”
5. “Either Aldo is Italian and Bob is English, or neither Aldo is Italian nor Bob is English”

Solution.

1. $\neg A$
2. $A \wedge B$
3. $A \rightarrow \neg B$
4. $A \vee (\neg A \rightarrow B)$ *logically equivalent to* $A \vee B$
5. $(A \wedge B) \vee (\neg A \wedge \neg B)$ *logically equivalent to* $A \leftrightarrow B$

*