Limits, Differentiation and IntegrationFormulae

Limits

$n^c x^n \to 0$ as $n \to \infty$ if $|x| < 1$ (any fixed c)

$x^n/n! \to 0$ as $n \to \infty$ (any fixed x)

$(1 + x/n)^n \to \mathrm{e}^x$ as $n \to \infty$, $x \ln x \to 0$ as $x \to 0$

If $f(a) = g(a) = 0$ then $\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{f'(a)}{g'(a)}$ (l'Hôpital's rule)

Differentiation

$$(uv)' = u'v + uv', \quad \left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2}$$

$$(uv)^{(n)} = u^{(n)}v + nu^{(n-1)}v^{(1)} + \cdots + {}^nC_r u^{(n-r)}v^{(r)} + \cdots + uv^{(n)}$$ Leibniz Theorem

where $^nC_r \equiv \binom{n}{r} = \frac{n!}{r!(n-r)!}$

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}(\sin x) &= \cos x & \frac{\mathrm{d}}{\mathrm{d}x}(\sinh x) &= \cosh x \\
\frac{\mathrm{d}}{\mathrm{d}x}(\cos x) &= -\sin x & \frac{\mathrm{d}}{\mathrm{d}x}(\cosh x) &= \sinh x \\
\frac{\mathrm{d}}{\mathrm{d}x}(\tan x) &= \sec^2 x & \frac{\mathrm{d}}{\mathrm{d}x}(\tanh x) &= \mathrm{sech}^2\, x \\
\frac{\mathrm{d}}{\mathrm{d}x}(\sec x) &= \sec x \tan x & \frac{\mathrm{d}}{\mathrm{d}x}(\mathrm{sech}\, x) &= -\mathrm{sech}\, x \tanh x \\
\frac{\mathrm{d}}{\mathrm{d}x}(\cot x) &= -\mathrm{cosec}^2\, x & \frac{\mathrm{d}}{\mathrm{d}x}(\coth x) &= -\mathrm{cosech}^2\, x \\
\frac{\mathrm{d}}{\mathrm{d}x}(\mathrm{cosec}\, x) &= -\mathrm{cosec}\, x \cot x & \frac{\mathrm{d}}{\mathrm{d}x}(\mathrm{cosech}\, x) &= -\mathrm{cosech}\, x \coth x
\end{aligned}$$

Integration

Standard forms

$$\int x^n \, \mathrm{d}x = \frac{x^{n+1}}{n+1} + c \qquad \text{for } n \neq -1$$

$$\int \frac{1}{x} \, \mathrm{d}x = \ln x + c \qquad \int \ln x \, \mathrm{d}x = x(\ln x - 1) + c$$

$$\int \mathrm{e}^{ax} \, \mathrm{d}x = \frac{1}{a}\mathrm{e}^{ax} + c \qquad \int x\,\mathrm{e}^{ax} \, \mathrm{d}x = \mathrm{e}^{ax}\left(\frac{x}{a} - \frac{1}{a^2}\right) + c$$

$$\int x \ln x \, \mathrm{d}x = \frac{x^2}{2}\left(\ln x - \frac{1}{2}\right) + c$$

$$\int \frac{1}{a^2 + x^2} \, \mathrm{d}x = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + c$$

$$\int \frac{1}{a^2 - x^2} \, \mathrm{d}x = \frac{1}{a}\tanh^{-1}\left(\frac{x}{a}\right) + c = \frac{1}{2a}\ln\left(\frac{a+x}{a-x}\right) + c \qquad \text{for } x^2 < a^2$$

$$\int \frac{1}{x^2 - a^2} \, \mathrm{d}x = -\frac{1}{a}\coth^{-1}\left(\frac{x}{a}\right) + c = \frac{1}{2a}\ln\left(\frac{x-a}{x+a}\right) + c \qquad \text{for } x^2 > a^2$$

$$\int \frac{x}{(x^2 \pm a^2)^n} \, \mathrm{d}x = \frac{-1}{2(n-1)}\frac{1}{(x^2 \pm a^2)^{n-1}} + c \qquad \text{for } n \neq 1$$

$$\int \frac{x}{x^2 \pm a^2} \, \mathrm{d}x = \frac{1}{2}\ln(x^2 \pm a^2) + c$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} \, \mathrm{d}x = \sin^{-1}\left(\frac{x}{a}\right) + c$$

$$\int \frac{1}{\sqrt{x^2 \pm a^2}} \, \mathrm{d}x = \ln\left(x + \sqrt{x^2 \pm a^2}\right) + c$$

$$\int \frac{x}{\sqrt{x^2 \pm a^2}} \, \mathrm{d}x = \sqrt{x^2 \pm a^2} + c$$

$$\int \sqrt{a^2 - x^2} \, \mathrm{d}x = \frac{1}{2}\left[x\sqrt{a^2 - x^2} + a^2 \sin^{-1}\left(\frac{x}{a}\right)\right] + c$$

$$\int_0^\infty \frac{1}{(1+x)x^p} \, \mathrm{d}x = \pi \operatorname{cosec} p\pi \qquad \text{for } p < 1$$

$$\int_0^\infty \cos(x^2)\, dx = \int_0^\infty \sin(x^2) \, \mathrm{d}x = \frac{1}{2}\sqrt{\frac{\pi}{2}}$$

$$\int_{-\infty}^\infty \exp(-x^2/2\sigma^2) \, \mathrm{d}x = \sigma\sqrt{2\pi}$$

$$\int_{-\infty}^\infty x^n \exp(-x^2/2\sigma^2) \, \mathrm{d}x = \begin{cases} 1 \times 3 \times 5 \times \cdots (n-1)\sigma^{n+1}\sqrt{2\pi} & \text{for } n \geq 2 \text{ and even} \\ 0 & \text{for } n \geq 1 \text{ and odd} \end{cases}$$

$$\int \sin mx \sin nx \, \mathrm{d}x = \frac{\sin(m-n)x}{2(m-n)} - \frac{\sin(m+n)x}{2(m+n)} + c \qquad \text{if } m^2 \neq n^2$$

$$\int \cos mx \cos nx \, \mathrm{d}x = \frac{\sin(m-n)x}{2(m-n)} + \frac{\sin(m+n)x}{2(m+n)} + c \qquad \text{if } m^2 \neq n^2$$

$$\begin{aligned}
\int \sin x \, \mathrm{d}x &= -\cos x + c & \int \sinh x \, \mathrm{d}x &= \cosh x + c \\
\int \cos x \, \mathrm{d}x &= \sin x + c & \int \cosh x \, \mathrm{d}x &= \sinh x + c \\
\int \tan x \, \mathrm{d}x &= -\ln(\cos x) + c & \int \tanh x \, \mathrm{d}x &= \ln(\cosh x) + c \\
\int \operatorname{cosec} x \, \mathrm{d}x &= \ln(\operatorname{cosec} x - \cot x) + c & \int \operatorname{cosech} x \, \mathrm{d}x &= \ln\left[\tanh(x/2)\right] + c \\
\int \sec x \, \mathrm{d}x &= \ln(\sec x + \tan x) + c & \int \operatorname{sech} x \, \mathrm{d}x &= 2\tan^{-1}(\mathrm{e}^x) + c \\
\int \cot x \, \mathrm{d}x &= \ln(\sin x) + c & \int \coth x \, \mathrm{d}x &= \ln(\sinh x) + c
\end{aligned}$$

Standard substitutions

If the integrand is a function of:	substitute:
$(a^2 - x^2)$ or $\sqrt{a^2 - x^2}$	$x = a\sin\theta$ or $x = a\cos\theta$
$(x^2 + a^2)$ or $\sqrt{x^2 + a^2}$	$x = a\tan\theta$ or $x = a\sinh\theta$
$(x^2 - a^2)$ or $\sqrt{x^2 - a^2}$	$x = a\sec\theta$ or $x = a\cosh\theta$

If the integrand is a rational function of $\sin x$ or $\cos x$ or both, substitute $t = \tan(x/2)$ and use the results:

$$\sin x = \frac{2t}{1+t^2} \qquad \cos x = \frac{1-t^2}{1+t^2} \qquad \mathrm{d}x = \frac{2\,\mathrm{d}t}{1+t^2}.$$

If the integrand is of the form:	substitute:
$\displaystyle\int \frac{\mathrm{d}x}{(ax+b)\sqrt{px+q}}$	$px + q = u^2$
$\displaystyle\int \frac{\mathrm{d}x}{(ax+b)\sqrt{px^2+qx+r}}$	$ax + b = \dfrac{1}{u}.$

Integration by parts

$$\int_a^b u \, \mathrm{d}v = uv\Big|_a^b - \int_a^b v \, \mathrm{d}u$$

Differentiation of an integral

If $f(x, \alpha)$ is a function of x containing a parameter α and the limits of integration a and b are functions of α then

$$\frac{\mathrm{d}}{\mathrm{d}\alpha} \int_{a(\alpha)}^{b(\alpha)} f(x, \alpha) \, \mathrm{d}x = f(b, \alpha)\frac{\mathrm{d}b}{\mathrm{d}\alpha} - f(a, \alpha)\frac{\mathrm{d}a}{\mathrm{d}\alpha} + \int_{a(\alpha)}^{b(\alpha)} \frac{\partial}{\partial\alpha} f(x, \alpha) \, \mathrm{d}x.$$

Special case,

$$\frac{\mathrm{d}}{\mathrm{d}x} \int_a^x f(y) \, \mathrm{d}y = f(x).$$

Dirac δ-'function'

$$\delta(t - \tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp[\mathrm{i}\omega(t - \tau)] \, \mathrm{d}\omega.$$

If $f(t)$ is an arbitrary function of t then $\int_{-\infty}^{\infty} \delta(t - \tau) f(t) \, \mathrm{d}t = f(\tau)$.

$\delta(t) = 0$ if $t \neq 0$, also $\int_{-\infty}^{\infty} \delta(t) \, \mathrm{d}t = 1$

Reduction formulae

Factorials

$n! = n(n-1)(n-2)\ldots1, \qquad 0! = 1.$

Stirling's formula for large n: $\ln(n!) \approx n \ln n - n.$

For any $p > -1$, $\int_0^\infty x^p \, \mathrm{e}^{-x} \, \mathrm{d}x = p \int_0^\infty x^{p-1} \, \mathrm{e}^{-x} \, \mathrm{d}x = p!. \qquad (-{}^1\!/_2)! = \sqrt{\pi}, \qquad ({}^1\!/_2)! = {}^{\sqrt{\pi}}\!/_2$, etc.

For any $p, q > -1$, $\int_0^1 x^p (1 - x)^q \, \mathrm{d}x = \frac{p!q!}{(p + q + 1)!}.$

Trigonometrical

If m, n are integers,

$$\int_0^{\pi/2} \sin^m\theta \, \cos^n\theta \, \mathrm{d}\theta = \frac{m-1}{m+n} \int_0^{\pi/2} \sin^{m-2}\theta \, \cos^n\theta \, \mathrm{d}\theta = \frac{n-1}{m+n} \int_0^{\pi/2} \sin^m\theta \, \cos^{n-2}\theta \, \mathrm{d}\theta$$

and can therefore be reduced eventually to one of the following integrals

$$\int_0^{\pi/2} \sin\theta \, \cos\theta \, \mathrm{d}\theta = \frac{1}{2}, \qquad \int_0^{\pi/2} \sin\theta \, \mathrm{d}\theta = 1, \qquad \int_0^{\pi/2} \cos\theta \, \mathrm{d}\theta = 1, \qquad \int_0^{\pi/2} \mathrm{d}\theta = \frac{\pi}{2}.$$

Other

If $I_n = \int_0^\infty x^n \exp(-\alpha x^2) \, \mathrm{d}x$ then $I_n = \frac{(n-1)}{2\alpha} I_{n-2}, \quad I_0 = \frac{1}{2}\sqrt{\frac{\pi}{\alpha}}, \quad I_1 = \frac{1}{2\alpha}.$