Integral Calculus Homework

Antidifferentiation - The Indefinite Integral

In problems 1 through 7, find the indicated integral.

1. $\int \sqrt{x}dx$
   Solution.
$$\int \sqrt{x}dx = \int x^{\frac{1}{2}}dx = \frac{2}{3}x^{\frac{3}{2}} + C = \frac{2}{3}x\sqrt{x} + C.$$

2. $\int 3e^x dx$
   Solution.
$$\int 3e^x dx = 3\int e^x dx = 3e^x + C.$$

3. $\int (3x^2 - \sqrt{5x} + 2)dx$
   Solution.
$$\begin{aligned}\int (3x^2 - \sqrt{5x} + 2)dx &= 3\int x^2 dx - \sqrt{5}\int \sqrt{x}dx + 2\int dx = \\ &= 3\cdot\frac{1}{3}x^3 - \sqrt{5}\cdot\frac{2}{3}x\sqrt{x} + 2x + C = \\ &= x^3 - \frac{2}{3}x\sqrt{5x} + 2x + C.\end{aligned}$$

4. $\int \left(\frac{1}{2x} - \frac{2}{x} + \frac{3}{\sqrt{x}}\right)dx$
   Solution.
$$\begin{aligned}\int \left(\frac{1}{2x} - \frac{2}{x^2} + \frac{3}{\sqrt{x}}\right)dx &= \frac{1}{2}\int \frac{1}{x}dx - 2\int x^{-2}dx + 3\int x^{-\frac{1}{2}}dx = \\ &= \frac{1}{2}\ln|x| - 2\cdot(-1)x^{-1} + 3\cdot 2x^{\frac{1}{2}} + C = \\ &= \frac{\ln|x|}{2} + \frac{2}{x} + 6\sqrt{x} + C.\end{aligned}$$

5. $\int \left(2e^x + \frac{6}{x} + \ln 2\right) dx$
Solution.

$$\begin{aligned}\int \left(2e^x + \frac{6}{x} + \ln 2\right) dx &= 2\int e^x dx + 6\int \frac{1}{x} dx + \ln 2 \int dx = \\ &= 2e^x + 6\ln|x| + (\ln 2)x + C.\end{aligned}$$

6. $\int \frac{x^2+3x-2}{\sqrt{x}} dx$
Solution.

$$\begin{aligned}\int \frac{x^2+3x-2}{\sqrt{x}} dx &= \int x^{\frac{3}{2}} dx + 3\int x^{\frac{1}{2}} dx - 2\int x^{-\frac{1}{2}} dx = \\ &= \frac{2}{5}x^{\frac{5}{2}} + 3\cdot\frac{2}{3}x^{\frac{3}{2}} - 2\cdot 2x^{\frac{1}{2}} + C = \\ &= \frac{2}{5}x^{\frac{5}{2}} + 2x^{\frac{3}{2}} - 4x^{\frac{1}{2}} + C = \\ &= \frac{2}{5}x^2\sqrt{x} + 2x\sqrt{x} - 4\sqrt{x} + C.\end{aligned}$$

7. $\int (x^3 - 2x^2)\left(\frac{1}{x} - 5\right) dx$
Solution.

$$\begin{aligned}\int (x^3 - 2x^2)\left(\frac{1}{x} - 5\right) dx &= \int (x^2 - 5x^3 - 2x + 10x^2) dx = \\ &= \int (-5x^3 + 11x^2 - 2x) dx = \\ &= -5\cdot\frac{1}{4}x^4 + 11\cdot\frac{1}{3}x^3 - 2\cdot\frac{1}{2}x^2 + C = \\ &= -\frac{5}{4}x^4 + \frac{11}{3}x^3 - x^2 + C.\end{aligned}$$

8. Find the function f whose tangent has slope $x^3 - \frac{2}{x^2} + 2$ for each value of x and whose graph passes through the point $(1, 3)$.
Solution. The slope of the tangent is the derivative of f. Thus

$$f'(x) = x^3 - \frac{2}{x^2} + 2$$

and so $f(x)$ is the indefinite integral

$$\begin{aligned}f(x) &= \int f'(x) dx = \int \left(x^3 - \frac{2}{x^2} + 2\right) dx = \\ &= \frac{1}{4}x^4 + \frac{2}{x} + 2x + C.\end{aligned}$$

Using the fact that the graph of f passes through the point $(1, 3)$ you get

$$3 = \frac{1}{4} + 2 + 2 + C \quad \text{or} \quad C = -\frac{5}{4}.$$

Therefore, the desired function is $f(x) = \frac{1}{4}x^4 + \frac{2}{x} + 2x - \frac{5}{4}$.

9. It is estimated that t years from now the population of a certain lakeside community will be changing at the rate of $0.6t^2 + 0.2t + 0.5$ thousand people per year. Environmentalists have found that the level of pollution in the lake increases at the rate of approximately 5 units per 1000 people. By how much will the pollution in the lake increase during the next 2 years?

   Solution. Let $P(t)$ denote the population of the community t years from now. Then the rate of change of the population with respect to time is the derivative

$$\frac{dP}{dt} = P'(t) = 0.6t^2 + 0.2t + 0.5.$$

   It follows that the population function $P(t)$ is an antiderivative of $0.6t^2 + 0.2t + 0.5$. That is,

$$\begin{aligned} P(t) &= \int P'(t)dt = \int (0.6t^2 + 0.2t + 0.5)dt = \\ &= 0.2t^3 + 0.1t^2 + 0.5t + C \end{aligned}$$

   for some constant C. During the next 2 years, the population will grow on behalf of

$$\begin{aligned} P(2) - P(0) &= 0.2 \cdot 2^3 + 0.1 \cdot 2^2 + 0.5 \cdot 2 + C - C = \\ &= 1.6 + 0.4 + 1 = 3 \text{ thousand people.} \end{aligned}$$

   Hence, the pollution in the lake will increase on behalf of $5 \cdot 3 = 15$ units.

10. An object is moving so that its speed after t minutes is $v(t) = 1+4t+3t^2$ meters per minute. How far does the object travel during 3rd minute?

   Solution. Let $s(t)$ denote the displacement of the car after t minutes. Since $v(t) = \frac{ds}{dt} = s'(t)$ it follows that

$$s(t) = \int v(t)dt = \int (1 + 4t + 3t^2)dt = t + 2t^2 + t^3 + C.$$

   During the 3rd minute, the object travels

$$\begin{aligned} s(3) - s(2) &= 3 + 2 \cdot 9 + 27 + C - 2 - 2 \cdot 4 - 8 - C = \\ &= 30 \text{ meters.} \end{aligned}$$

Integration by Substitution

In problems 1 through 8, find the indicated integral.

1. $\int (2x+6)^5 dx$
Solution. Substituting $u = 2x+6$ and $\frac{1}{2}du = dx$, you get

$$\int (2x+6)^5 dx = \frac{1}{2}\int u^5 du = \frac{1}{12}u^6 + C = \frac{1}{12}(2x+6)^6 + C.$$

2. $\int \left[(x-1)^5 + 3(x-1)^2 + 5\right] dx$
Solution. Substituting $u = x-1$ and $du = dx$, you get

$$\begin{aligned}\int \left[(x-1)^5 + 3(x-1)^2 + 5\right] dx &= \int (u^5 + 3u^2 + 5)du = \\ &= \frac{1}{6}u^6 + u^3 + 5u + C = \\ &= \frac{1}{6}(x-1)^6 + (x-1)^3 + 5(x-1) + C.\end{aligned}$$

Since, for a constant C, $C-5$ is again a constant, you can write

$$\int \left[(x-1)^5 + 3(x-1)^2 + 5\right] dx = \frac{1}{6}(x-1)^6 + (x-1)^3 + 5x + C.$$

3. $\int xe^{x^2} dx$
Solution. Substituting $u = x^2$ and $\frac{1}{2}du = xdx$, you get

$$\int xe^{x^2} dx = \frac{1}{2}\int e^u du = \frac{1}{2}e^u + C = \frac{1}{2}e^{x^2} + C.$$

4. $\int x^5 e^{1-x^6} dx$
Solution. Substituting $u = 1-x^6$ and $-\frac{1}{6}du = x^5 dx$, you get

$$\int x^5 e^{1-x^6} dx = -\frac{1}{6}\int e^u du = -\frac{1}{6}e^u + C = -\frac{1}{6}e^{1-x^6} + C.$$

5. $\int \frac{2x^4}{x^5+1} dx$
Solution. Substituting $u = x^5+1$ and $\frac{2}{5}du = 2x^4 dx$, you get

$$\int \frac{2x^4}{x^5+1} dx = \frac{2}{5}\int \frac{1}{u} du = \frac{2}{5}\ln|u| + C = \frac{2}{5}\ln|x^5+1| + C.$$

6. $\int \frac{10x^3-5x}{\sqrt{x^4-x^2+6}}dx$
Solution. Substituting $u = x^4 - x^2 + 6$ and $\frac{5}{2}du = (10x^3 - 5x)dx$, you get

$$\begin{aligned}\int \frac{10x^3-5x}{\sqrt{x^4-x^2+6}}dx &= \frac{5}{2}\int \frac{1}{\sqrt{u}}du = \frac{5}{2}\int u^{-\frac{1}{2}}du = \frac{5}{2}\cdot 2u^{\frac{1}{2}} + C = \\ &= 5\sqrt{x^4-x^2+6} + C.\end{aligned}$$

7. $\int \frac{1}{x\ln x}dx$
Solution. Substituting $u = \ln x$ and $du = \frac{1}{x}dx$, you get

$$\int \frac{1}{x\ln x}dx = \int \frac{1}{u}du = \ln|u| + C = \ln|\ln x| + C.$$

8. $\int \frac{\ln x^2}{x}dx$
Solution. Substituting $u = \ln x$ and $du = \frac{1}{x}dx$, you get

$$\int \frac{\ln x^2}{x}dx = \int \frac{2\ln x}{x}dx = 2\int u du = 2\cdot\frac{1}{2}u^2 + C = (\ln x)^2 + C.$$

9. Use an appropriate change of variables to find the integral

$$\int (x+1)(x-2)^9 dx.$$

Solution. Substituting $u = x - 2$, $u + 3 = x + 1$ and $du = dx$, you get

$$\begin{aligned}\int (x+1)(x-2)^9 dx &= \int (u+3)u^9 du = \int (u^{10} + 3u^9)du = \\ &= \frac{1}{11}u^{11} + \frac{3}{10}u^{10} + C = \\ &= \frac{1}{11}(x-2)^{11} + \frac{3}{10}(x-2)^{10} + C.\end{aligned}$$

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Integration by Parts

In problems 1 through 8, use integration by parts to find the given integral.

1. $\int xe^{0.1x}dx$
Solution. Since the factor $e^{0.1x}$ is easy to integrate and the factor x is simplified by differentiation, try integration by parts with

$$g(x) = e^{0.1x} \qquad \text{and} \qquad f(x) = x.$$

Then,

$$G(x) = \int e^{0.1x}dx = 10e^{0.1x} \qquad \text{and} \qquad f'(x) = 1$$

and so

$$\begin{aligned} \int xe^{0.1x}dx &= 10xe^{0.1x} - 10\int e^{0.1x}dx = 10xe^{0.1x} - 100e^{0.1x} + C = \\ &= 10(x-10)e^{0.1x} + C. \end{aligned}$$

2. $\int (3-2x)e^{-x}dx$
Solution. Since the factor e^{-x} is easy to integrate and the factor $3-2x$ is simplified by differentiation, try integration by parts with

$$g(x) = e^{-x} \qquad \text{and} \qquad f(x) = 3-2x.$$

Then,

$$G(x) = \int e^{-x}dx = -e^{-x} \qquad \text{and} \qquad f'(x) = -2$$

and so

$$\begin{aligned} \int (3-2x)e^{-x}dx &= (3-2x)(-e^{-x}) - 2\int e^{-x}dx = \\ &= (2x-3)e^{-x} + 2e^{-x} + C = (2x-1)e^{-x} + C. \end{aligned}$$

3. $\int x\ln x^2dx$
Solution. In this case, the factor x is easy to integrate, while the factor $\ln x^2$ is simplified by differentiation. This suggests that you try integration by parts with

$$g(x) = x \qquad \text{and} \qquad f(x) = \ln x^2.$$

Then,

$$G(x) = \int xdx = \frac{1}{2}x^2 \qquad \text{and} \qquad f'(x) = \frac{1}{x^2}2x = \frac{2}{x}$$

and so

$$\int x\ln x^2 dx = \frac{1}{2}x^2\ln x^2 - \int \frac{1}{2}x^2\frac{2}{x}dx = \frac{1}{2}x^2\ln x^2 - \int x dx =$$
$$= \frac{1}{2}x^2\ln x^2 - \frac{1}{2}x^2 + C = \frac{1}{2}x^2\left(\ln x^2 - 1\right) + C.$$

4. $\int x\sqrt{1-x}dx$
Solution. Since the factor $\sqrt{1-x}$ is easy to integrate and the factor x is simplified by differentiation, try integration by parts with

$$g(x) = \sqrt{1-x} \quad \text{and} \quad f(x) = x.$$

Then,

$$G(x) = \int \sqrt{1-x}dx = -\frac{2}{3}(1-x)^{\frac{3}{2}} \quad \text{and} \quad f'(x) = 1$$

and so

$$\int x\sqrt{1-x}dx = -\frac{2}{3}x(1-x)^{\frac{3}{2}} + \frac{2}{3}\int (1-x)^{\frac{3}{2}}dx =$$
$$= -\frac{2}{3}x(1-x)^{\frac{3}{2}} + \frac{2}{3}\left(-\frac{2}{5}(1-x)^{\frac{5}{2}}\right) + C =$$
$$= -\frac{2}{3}x(1-x)^{\frac{3}{2}} - \frac{4}{15}(1-x)^{\frac{5}{2}} + C =$$
$$= -\frac{2}{3}x(1-x)\sqrt{1-x} - \frac{4}{15}(1-x)^2\sqrt{1-x} + C.$$

5. $\int (x+1)(x+2)^6 dx$
Solution. Since the factor $(x+2)^6$ is easy to integrate and the factor $x+1$ is simplified by differentiation, try integration by parts with

$$g(x) = (x+2)^6 \quad \text{and} \quad f(x) = x+1.$$

Then,

$$G(x) = \int (x+2)^6 dx = \frac{1}{7}(x+2)^7 \quad \text{and} \quad f'(x) = 1$$

and so

$$\int (x+1)(x+2)^6 dx = \frac{1}{7}(x+1)(x+2)^7 - \frac{1}{7}\int (x+2)^7 dx =$$
$$= \frac{1}{7}(x+1)(x+2)^7 - \frac{1}{7}\frac{1}{8}(x+2)^8 + C =$$
$$= \frac{1}{56}\left[8(x+1) - (x+2)\right](x+2)^7 + C =$$
$$= \frac{1}{56}(7x+6)(x+2)^7 + C.$$

6. $\int x^3 e^{2x} dx$

Solution. Since the factor e^{2x} is easy to integrate and the factor x^3 is simplified by differentiation, try integration by parts with

$$g(x) = e^{2x} \quad \text{and} \quad f(x) = x^3.$$

Then,

$$G(x) = \int e^{2x} dx = \frac{1}{2} e^{2x} \quad \text{and} \quad f'(x) = 3x^2$$

and so

$$\int x^3 e^{2x} dx = \frac{1}{2} x^3 e^{2x} - \frac{3}{2} \int x^2 e^{2x} dx.$$

To find $\int x^2 e^{2x} dx$, you have to integrate by parts again, but this time with

$$g(x) = e^{2x} \quad \text{and} \quad f(x) = x^2.$$

Then,

$$G(x) = \frac{1}{2} e^{2x} \quad \text{and} \quad f'(x) = 2x$$

and so

$$\int x^2 e^{2x} dx = \frac{1}{2} x^2 e^{2x} - \int x e^{2x} dx.$$

To find $\int x e^{2x} dx$, you have to integrate by parts once again, this time with

$$g(x) = e^{2x} \quad \text{and} \quad f(x) = x.$$

Then,

$$G(x) = \frac{1}{2} e^{2x} \quad \text{and} \quad f'(x) = 1$$

and so

$$\int x e^{2x} dx = \frac{1}{2} x e^{2x} - \frac{1}{2} \int e^{2x} dx = \frac{1}{2} x e^{2x} - \frac{1}{4} e^{2x}.$$

Finally,

$$\begin{aligned} \int x^3 e^{2x} dx &= \frac{1}{2} x^3 e^{2x} - \frac{3}{2} \left[\frac{1}{2} x^2 e^{2x} - \left(\frac{1}{2} x e^{2x} - \frac{1}{4} e^{2x} \right) \right] + C = \\ &= \left(\frac{1}{2} x^3 - \frac{3}{4} x^2 + \frac{3}{4} x - \frac{3}{8} \right) e^{2x} + C. \end{aligned}$$

7. $\int \frac{\ln x}{x^3} dx$

Solution. In this case, the factor $\frac{1}{x^3}$ is easy to integrate, while the factor $\ln x$ is simplified by differentiation. This suggests that you try integration by parts with

$$g(x) = \frac{1}{x^3} \quad \text{and} \quad f(x) = \ln x.$$

Then,

$$G(x) = \int \frac{1}{x^3} dx = -\frac{1}{2}x^{-2} = -\frac{1}{2x^2} \quad \text{and} \quad f'(x) = \frac{1}{x}$$

and so

$$\begin{aligned} \int \frac{\ln x}{x^3} dx &= -\frac{\ln x}{2x^2} + \frac{1}{2}\int \frac{1}{x^3} dx = -\frac{\ln x}{2x^2} + \frac{1}{2}\left(-\frac{1}{2x^2}\right) + C = \\ &= -\frac{\ln x}{2x^2} - \frac{1}{4x^2} + C. \end{aligned}$$

8. $\int x^3 e^{x^2} dx$

Solution. First rewrite the integrand as $x^2\left(xe^{x^2}\right)$, and then integrate by parts with

$$g(x) = xe^{x^2} \quad \text{and} \quad f(x) = x^2.$$

Then, from Exercise 6.2.3 you get

$$G(x) = \int xe^{x^2} dx = \frac{1}{2}e^{x^2} \quad \text{and} \quad f'(x) = 2x$$

and so

$$\begin{aligned} \int x^3 e^{x^2} dx &= \frac{1}{2}x^2 e^{x^2} - \int xe^{x^2} = \frac{1}{2}x^2 e^{x^2} - \frac{1}{2}e^{x^2} + C = \\ &= \frac{1}{2}(x^2 - 1)e^{x^2} + C. \end{aligned}$$