Functions of Several Variables Formulae

If $\phi = f(x, y, z, \ldots)$ then $\dfrac{\partial \phi}{\partial x}$ implies differentiation with respect to x keeping $y, z, \ldots$ constant.

$$d\phi = \frac{\partial \phi}{\partial x}dx + \frac{\partial \phi}{\partial y}dy + \frac{\partial \phi}{\partial z}dz + \cdots \quad \text{and} \quad \delta\phi \approx \frac{\partial \phi}{\partial x}\delta x + \frac{\partial \phi}{\partial y}\delta y + \frac{\partial \phi}{\partial z}\delta z + \cdots$$

where $x, y, z, \ldots$ are independent variables. $\dfrac{\partial \phi}{\partial x}$ is also written as $\left(\dfrac{\partial \phi}{\partial x}\right)_{y,\ldots}$ or $\left.\dfrac{\partial \phi}{\partial x}\right|_{y,\ldots}$ when the variables kept constant need to be stated explicitly.

If ϕ is a well-behaved function then $\dfrac{\partial^2 \phi}{\partial x\,\partial y} = \dfrac{\partial^2 \phi}{\partial y\,\partial x}$ etc.

If $\phi = f(x, y)$,

$$\left(\frac{\partial \phi}{\partial x}\right)_y = \frac{1}{\left(\dfrac{\partial x}{\partial \phi}\right)_y}, \qquad \left(\frac{\partial \phi}{\partial x}\right)_y \left(\frac{\partial x}{\partial y}\right)_\phi \left(\frac{\partial y}{\partial \phi}\right)_x = -1.$$

Taylor series for two variables

If $\phi(x, y)$ is well-behaved in the vicinity of $x = a, y = b$ then it has a Taylor series

$$\phi(x, y) = \phi(a + u, b + v) = \phi(a, b) + u\frac{\partial \phi}{\partial x} + v\frac{\partial \phi}{\partial y} + \frac{1}{2!}\left(u^2\frac{\partial^2 \phi}{\partial x^2} + 2uv\frac{\partial^2 \phi}{\partial x\,\partial y} + v^2\frac{\partial^2 \phi}{\partial y^2}\right) + \cdots$$

where $x = a + u, y = b + v$ and the differential coefficients are evaluated at $x = a, \quad y = b$

Stationary points

A function $\phi = f(x, y)$ has a stationary point when $\dfrac{\partial \phi}{\partial x} = \dfrac{\partial \phi}{\partial y} = 0$. Unless $\dfrac{\partial^2 \phi}{\partial x^2} = \dfrac{\partial^2 \phi}{\partial y^2} = \dfrac{\partial^2 \phi}{\partial x\,\partial y} = 0$, the following conditions determine whether it is a minimum, a maximum or a saddle point.

$$\left.\begin{array}{ll} \text{Minimum:} & \dfrac{\partial^2 \phi}{\partial x^2} > 0, \text{ or } \dfrac{\partial^2 \phi}{\partial y^2} > 0, \\ \text{Maximum:} & \dfrac{\partial^2 \phi}{\partial x^2} < 0, \text{ or } \dfrac{\partial^2 \phi}{\partial y^2} < 0, \end{array}\right\} \quad \text{and} \quad \frac{\partial^2 \phi}{\partial x^2}\frac{\partial^2 \phi}{\partial y^2} > \left(\frac{\partial^2 \phi}{\partial x\,\partial y}\right)^2$$

$$\text{Saddle point:} \quad \frac{\partial^2 \phi}{\partial x^2}\frac{\partial^2 \phi}{\partial y^2} < \left(\frac{\partial^2 \phi}{\partial x\,\partial y}\right)^2$$

If $\dfrac{\partial^2 \phi}{\partial x^2} = \dfrac{\partial^2 \phi}{\partial y^2} = \dfrac{\partial^2 \phi}{\partial x\,\partial y} = 0$ the character of the turning point is determined by the next higher derivative.

Changing variables: the chain rule

If $\phi = f(x, y, \ldots)$ and the variables $x, y, \ldots$ are functions of independent variables $u, v, \ldots$ then

$$\frac{\partial \phi}{\partial u} = \frac{\partial \phi}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial \phi}{\partial y}\frac{\partial y}{\partial u} + \cdots$$

$$\frac{\partial \phi}{\partial v} = \frac{\partial \phi}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial \phi}{\partial y}\frac{\partial y}{\partial v} + \cdots$$

etc.

Changing variables in surface and volume integrals – Jacobians

If an area A in the x, y plane maps into an area A' in the u, v plane then

$$\int_A f(x,y)\,\mathrm{d}x\,\mathrm{d}y = \int_{A'} f(u,v)J\,\mathrm{d}u\,\mathrm{d}v \quad \text{where} \quad J = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix}$$

The Jacobian J is also written as $\dfrac{\partial(x,y)}{\partial(u,v)}$. The corresponding formula for volume integrals is

$$\int_V f(x,y,z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \int_{V'} f(u,v,w)J\,\mathrm{d}u\,\mathrm{d}v\,\mathrm{d}w \qquad \text{where now} \qquad J = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} & \dfrac{\partial x}{\partial w} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} & \dfrac{\partial y}{\partial w} \\ \dfrac{\partial z}{\partial u} & \dfrac{\partial z}{\partial v} & \dfrac{\partial z}{\partial w} \end{vmatrix}$$