Differential Equations Formulae

Diffusion (conduction) equation

$$\frac{\partial\psi}{\partial t} = \kappa\nabla^2\psi$$

Wave equation

$$\nabla^2\psi = \frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2}$$

Legendre's equation

$$(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + l(l+1)y = 0,$$

solutions of which are Legendre polynomials $P_l(x)$, where $P_l(x) = \frac{1}{2^l l!}\left(\frac{d}{dx}\right)^l (x^2-1)^l$, *Rodrigues' formula* so $P_0(x) = 1$, $P_1(x) = x$, $P_2(x) = \frac{1}{2}(3x^2-1)$ etc.

Recursion relation

$$P_l(x) = \frac{1}{l}\left[(2l-1)xP_{l-1}(x) - (l-1)P_{l-2}(x)\right]$$

Orthogonality

$$\int_{-1}^{1} P_l(x)P_{l'}(x)\,dx = \frac{2}{2l+1}\delta_{ll'}$$

Bessel's equation

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2-m^2)y = 0,$$

solutions of which are Bessel functions $J_m(x)$ of order m.

Series form of Bessel functions of the first kind

$$J_m(x) = \sum_{k=0}^{\infty}\frac{(-1)^k(x/2)^{m+2k}}{k!(m+k)!} \qquad \text{(integer } m\text{)}.$$

The same general form holds for non-integer $m > 0$.

Laplace's equation

$$\nabla^2 u = 0$$

If expressed in two-dimensional polar coordinates (see section 4), a solution is

$$u(\rho, \varphi) = \left[A\rho^n + B\rho^{-n}\right]\left[C\exp(in\varphi) + D\exp(-in\varphi)\right]$$

where A, B, C, D are constants and n is a real integer.

If expressed in three-dimensional polar coordinates (see section 4) a solution is

$$u(r, \theta, \varphi) = \left[Ar^l + Br^{-(l+1)}\right]P_l^m\left[C\sin m\varphi + D\cos m\varphi\right]$$

where l and m are integers with $l \geq |m| \geq 0$; A, B, C, D are constants;

$$P_l^m(\cos\theta) = \sin^{|m|}\theta\left[\frac{\mathrm{d}}{\mathrm{d}(\cos\theta)}\right]^{|m|} P_l(\cos\theta)$$

is the associated Legendre polynomial.

$$P_l^0(1) = 1.$$

If expressed in cylindrical polar coordinates (see section 4), a solution is

$$u(\rho, \varphi, z) = J_m(n\rho)\left[A\cos m\varphi + B\sin m\varphi\right]\left[C\exp(nz) + D\exp(-nz)\right]$$

where m and n are integers; A, B, C, D are constants.

Spherical harmonics

The normalized solutions $Y_l^m(\theta, \varphi)$ of the equation

$$\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\right]Y_l^m + l(l+1)Y_l^m = 0$$

are called spherical harmonics, and have values given by

$$Y_l^m(\theta, \varphi) = \sqrt{\frac{2l+1}{4\pi}\frac{(l-|m|)!}{(l+|m|)!}}P_l^m(\cos\theta)\ \mathrm{e}^{\mathrm{i}m\varphi} \times \begin{cases}(-1)^m & \text{for } m \geq 0\\ 1 & \text{for } m < 0\end{cases}$$

i.e., $Y_0^0 = \sqrt{\frac{1}{4\pi}}$, $\quad Y_1^0 = \sqrt{\frac{3}{4\pi}}\cos\theta$, $\quad Y_1^{\pm 1} = \mp\sqrt{\frac{3}{8\pi}}\sin\theta\ \mathrm{e}^{\pm\mathrm{i}\varphi}$, etc.

Orthogonality

$$\int_{4\pi} Y_l^{*m} Y_{l'}^{m'}\ \mathrm{d}\Omega = \delta_{ll'}\delta_{mm'}$$