Differential Equations Homework Help

Some differential equations can be made simpler by substituting a particular form of solution. Often you will be given a substitution to try.

In the special case of *homogeneous*[1] equations, there is one method of substitution that always works.

In the current context we are using *homogeneous* to mean a function of x and y that is left unchanged by multiplying both arguments by a constant, i.e.

$$f(x,y) = f(kx,ky).$$

If this is the case, we can solve homogeneous differential equations of the following form:

$$\frac{dy}{dx} = f(x,y).$$

A homogeneous differential equation can be solved by the substitution $y = xv$, where $v = v(x)$ is a function of x.

Example 1

Q)
Solve the differential equation:

$$\frac{dy}{dx} = \frac{x^2 + y^2}{2x^2}.$$

A)
Here we have not been told that the question is to be solved by substitution or what substitution to try. In general it would be worth thinking about solving the equation by simple separation or integrating factors first. Here it would be clear after a few moments of work that the equation can not be separated. As there is a y^2 on the RHS it can not be put in the form of an integrating factor problem either.

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[1] Calling these equations homogeneous can lead to some difficulty as you may have come across other differential equations referred to as homogeneous before and the two types bear little to no relation to each other!

So let's check to see if the RHS is homogeneous. First we identify $f(x,y)$ as:

$$f(x,y) = \frac{x^2 + y^2}{2x^2}.$$

For this function to be homogenous we require, for any choice of k, that:

$$f(x,y) = f(kx,ky).$$

Multiplying both arguments of $f(x,y)$ by k we find:

$$f(kx,ky) = \frac{(kx)^2 + (ky)^2}{2(kx)^2}.$$

Using the laws of indices to get rid of the brackets, we find:

$$f(kx,ky) = \frac{k^2x^2 + k^2y^2}{2k^2x^2},$$

now we may pull a factor of k^2 out of a set of brackets on the top of the fraction:

$$f(kx,ky) = \frac{k^2(x^2 + y^2)}{2k^2x^2}.$$

We can now see we have a factor of k^2 on both top and bottom; this can be cancelled off to leave:

$$f(kx,ky) = \frac{x^2 + y^2}{2x^2} = f(x,y).$$

We have shown that $f(kx,ky) = f(x,y)$, therefore the differential equation we were given is homogeneous. As we have a homogeneous equation we will attempt a substitution of the form $y = xv$. We need to differentiate the substitution to find the LHS of the original equation:

$$\frac{dy}{dx} = \frac{d}{dx}xv = v + x\frac{dv}{dx},$$

(we have used the product rule). Using this along with $y = xv$, we can rewrite the original equation as:

$$v + x\frac{dv}{dx} = \frac{x^2 + (xv)^2}{2x^2}.$$

On multiplying out the above equation we see that there is a common factor of x^2 on the top and bottom of the RHS,

$$v + x\frac{dv}{dx} = \frac{x^2 + x^2v^2}{2x^2},$$

therefore we can cancel this down to:

$$v + x\frac{dv}{dx} = \frac{1+v^2}{2}.$$

A cancellation such as this will always happen and the remaining equation should be solvable by a simple method such as separation or integrating factor.

The above equation is solvable by separation. The task of separating the equation is not general, so for this example we proceed quickly, only recording the major steps. Taking all the terms containing v onto the RHS we get:

$$x\frac{dv}{dx} = \frac{1+v^2}{2} - v,$$

and simplifying by multiplying both sides by two:

$$2x\frac{dv}{dx} = 1 + v^2 - 2v.$$

The RHS can be factored:

$$2x\frac{dv}{dx} = (1-v)^2,$$

Before separating the equation and integrating:

$$\int (1-v)^{-2}\, dv = \frac{1}{2}\int \frac{1}{x}\, dx.$$

Therefore we form the answer:

$$-\frac{1}{1-v} = \frac{1}{2}\ln(x) + C.$$

This is an equation in v and we want an equation in x so we now substitute back using $v = \frac{y}{x}$ to give:

$$-\frac{1}{1-\frac{y}{x}} = \frac{1}{2}\ln(x) + C.$$

Finally, this can be rearranged to find y as a function of x:

$$\ln(D) = C,$$
$$\frac{1}{\frac{y}{x}-1} = \ln(Dx^2),$$
$$\frac{y}{x}-1 = \frac{1}{\ln(Dx^2)},$$
$$\frac{y}{x} = 1+\frac{1}{\ln(Dx^2)},$$
$$y = x\left(1+\frac{1}{\ln(Dx^2)}\right).$$

Example 2

Q)
Sometimes we will be given a substitution to try. As an example, we may be given an equation:

$$x\frac{dy}{dx} + y = xy^2,$$

along with a suggestion of a substitution, $y = \frac{1}{v}$.

A)
As always the first step in the substitution is to work out what to replace the differential dy/dx with. To do this let us differentiate the substitution given:

$$y = \frac{1}{v},$$
$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{1}{v}\right).$$

This can be done by use of the chain rule:

$$\frac{dy}{dx} = \frac{dv}{dx}\cdot\frac{d}{dv}\left(\frac{1}{v}\right),$$

giving:

$$\frac{dy}{dx} = -\frac{1}{v^2}\frac{dv}{dx}.$$

Substituting this into the differential equation given in the question we find:

$$x\left(-\frac{1}{v^2}\frac{dv}{dx}\right)+\frac{1}{v}=x\left(\frac{1}{v}\right)^2.$$

The whole point of a substitution is that after doing it there must be a cancellation that allows us to solve the problem. Here we can see that we can multiply both sides by v^2, leaving us with:

$$-x\frac{dv}{dx}+v=x,$$

which can be rearranged into the form of equation that can be solved with an integrating factor. Dividing by $-x$ we find:

$$\frac{dv}{dx}-\frac{1}{x}v=-1.$$

Now we can form an integrating factor, first integrating:

$$F(x)=\int-\frac{1}{x}dx=-\ln(x),$$

and then exponentiating to give:

$$e^{F(x)}=e^{-\ln(x)}=\frac{1}{x},$$

where we have used the laws of logarithms to remove the negative at the front of the natural logarithm and to cancel out the exponentiation. Multiplying both sides of the differential equation by this integrating factor:

$$\frac{1}{x}\frac{dv}{dx}-\frac{1}{x^2}v=-\frac{1}{x},$$

the LHS can now be written as:

$$\frac{1}{x}\frac{dv}{dx}-\frac{1}{x^2}v=\frac{d}{dx}\left(\frac{1}{x}v\right),$$

giving us the problem as:

$$\frac{d}{dx}\left(\frac{1}{x}v\right)=-\frac{1}{x}.$$

Integrating both sides gives:

$$\frac{1}{x}v = -\ln(x) + C.$$

Once again we want to write the solution in terms of y therefore we use the inverse substitution $v = \frac{1}{y}$, obtained by rearranging the substitution we were given; this gives:

$$\frac{1}{xy} = -\ln(x) + C,$$

and we can solve for y:

$$\frac{xy}{1} = \frac{1}{-\ln(x) + C},$$

$$xy = \frac{1}{C - \ln(x)},$$

$$y = \frac{1}{x(C - \ln(x))}.$$