

## Solution 1 and 2

$$\text{Given } x_1 + 2x_2 + 3x_3 = 1,$$

$$-3x_1 + 2x_2 + x_3 = 2,$$

$$2x_1 - x_3 = 0,$$

$$3x_1 - x_2 + 2x_3 = 4.$$

(2) Find  $A = QR$ :

$$\text{From the system, Here } A = \begin{pmatrix} 1 & 2 & 3 \\ -3 & 2 & 1 \\ 2 & 0 & -1 \\ 3 & -1 & 2 \end{pmatrix}, X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, b = \begin{pmatrix} 1 \\ 2 \\ 0 \\ 4 \end{pmatrix}$$

$$\text{Here } a_1 = \begin{pmatrix} 1 \\ -3 \\ 2 \\ 3 \end{pmatrix}, a_2 = \begin{pmatrix} 2 \\ 2 \\ 0 \\ -1 \end{pmatrix}, a_3 = \begin{pmatrix} 3 \\ 1 \\ -1 \\ 2 \end{pmatrix}$$

Performing the Gram-Schmidt procedure, we get

$$u_1 = a_1 = \begin{pmatrix} 1 \\ -3 \\ 2 \\ 3 \end{pmatrix}$$

$$e_1 = \frac{u_1}{\|u_1\|} = \frac{1}{\sqrt{1+9+4+9}} \begin{pmatrix} 1 \\ -3 \\ 2 \\ 3 \end{pmatrix} = \frac{1}{\sqrt{23}} \begin{pmatrix} 1 \\ -3 \\ 2 \\ 3 \end{pmatrix}$$

$$u_2 = a_2 - (a_2 e_1) e_1 = \begin{pmatrix} 2 \\ 2 \\ 0 \\ -1 \end{pmatrix} - \left( \begin{pmatrix} 2 \\ 2 \\ 0 \\ -1 \end{pmatrix} \frac{1}{\sqrt{23}} \begin{pmatrix} 1 \\ -3 \\ 2 \\ 3 \end{pmatrix} \right) \frac{1}{\sqrt{23}} \begin{pmatrix} 1 \\ -3 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ 0 \\ -1 \end{pmatrix} + \left( \frac{7}{\sqrt{23}} \right) \frac{1}{\sqrt{23}} \begin{pmatrix} 1 \\ -3 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 53/23 \\ 25/23 \\ 14/23 \\ -2/23 \end{pmatrix}$$

$$e_2 = \frac{2u_2}{\|u_2\|} = \frac{1}{\sqrt{\left(\frac{53}{23}\right)^2 + \left(\frac{25}{23}\right)^2 + \left(\frac{14}{23}\right)^2 + \left(-\frac{2}{23}\right)^2}} \begin{pmatrix} 53/23 \\ 25/23 \\ 14/23 \\ -2/23 \end{pmatrix} = \begin{pmatrix} \frac{53}{3634} \sqrt{158} \sqrt{23} \\ \frac{25}{3634} \sqrt{158} \sqrt{23} \\ \frac{7}{1817} \sqrt{158} \sqrt{23} \\ -\frac{1}{1817} \sqrt{158} \sqrt{23} \end{pmatrix}$$

$$u_3 = a_3 - (a_3 e_1) e_1 - (a_3 e_2) e_2 = \begin{pmatrix} 3 \\ 1 \\ -1 \\ 2 \end{pmatrix} - \left( \begin{pmatrix} 3 \\ 1 \\ -1 \\ 2 \end{pmatrix} \frac{1}{\sqrt{23}} \begin{pmatrix} 1 \\ -3 \\ 2 \\ 3 \end{pmatrix} \right) - \left( \frac{1}{\sqrt{23}} \begin{pmatrix} 1 \\ -3 \\ 2 \\ 3 \end{pmatrix} \right) - \begin{pmatrix} 3 \\ 1 \\ -1 \\ 2 \end{pmatrix} - \begin{pmatrix} \frac{53}{3634} \sqrt{158} \sqrt{23} \\ \frac{25}{3634} \sqrt{158} \sqrt{23} \\ \frac{7}{1817} \sqrt{158} \sqrt{23} \\ -\frac{1}{1817} \sqrt{158} \sqrt{23} \end{pmatrix} - \begin{pmatrix} \frac{53}{3634} \sqrt{158} \sqrt{23} \\ \frac{25}{3634} \sqrt{158} \sqrt{23} \\ \frac{7}{1817} \sqrt{158} \sqrt{23} \\ -\frac{1}{1817} \sqrt{158} \sqrt{23} \end{pmatrix}$$

$$= \begin{pmatrix} 32/79 \\ 30/79 \\ -157/79 \\ 124/79 \end{pmatrix}$$

$$e_3 = \frac{u_3}{\|u_3\|} = \frac{1}{\sqrt{\left(\frac{32}{79}\right)^2 + \left(\frac{30}{79}\right)^2 + \left(-\frac{157}{79}\right)^2 + \left(\frac{124}{79}\right)^2}} \begin{pmatrix} 32/79 \\ 30/79 \\ -157/79 \\ 124/79 \end{pmatrix} = \begin{pmatrix} \frac{32}{41949} \sqrt{531} \sqrt{79} \\ \frac{10}{13983} \sqrt{531} \sqrt{79} \\ -\frac{157}{41949} \sqrt{531} \sqrt{79} \\ \frac{124}{41949} \sqrt{531} \sqrt{79} \end{pmatrix}$$

$$\text{Thus } Q = (e_1 \ e_2 \ e_3) = \begin{pmatrix} \frac{1}{23} \sqrt{23} & \frac{53}{3634} \sqrt{3634} & \frac{32}{13983} \sqrt{4661} \\ -\frac{3}{23} \sqrt{23} & \frac{25}{3634} \sqrt{3634} & \frac{10}{4661} \sqrt{4661} \\ \frac{2}{23} \sqrt{23} & \frac{7}{1817} \sqrt{3634} & -\frac{157}{13983} \sqrt{4661} \\ \frac{3}{23} \sqrt{23} & -\frac{1}{1817} \sqrt{3634} & \frac{124}{13983} \sqrt{4661} \end{pmatrix}$$

$$a_1 e_1 = \begin{pmatrix} 1 \\ -3 \\ 2 \\ 3 \end{pmatrix} \frac{1}{\sqrt{23}} \begin{pmatrix} 1 \\ -3 \\ 2 \\ 3 \end{pmatrix} = \sqrt{23}, a_2 e_1 = \begin{pmatrix} 2 \\ 2 \\ 0 \\ -1 \end{pmatrix} \frac{1}{\sqrt{23}} \begin{pmatrix} 1 \\ -3 \\ 2 \\ 3 \end{pmatrix} = -\frac{7}{\sqrt{23}}, a_3 e_1 = \begin{pmatrix} 3 \\ 1 \\ -1 \\ 2 \end{pmatrix} \frac{1}{\sqrt{23}} \begin{pmatrix} 1 \\ -3 \\ 2 \\ 3 \end{pmatrix} = \frac{4}{\sqrt{23}}$$

$$a_2 e_2 = \begin{pmatrix} 2 \\ 2 \\ 0 \\ -1 \end{pmatrix} \begin{pmatrix} \frac{53}{3634} \sqrt{158} \sqrt{23} \\ \frac{25}{3634} \sqrt{158} \sqrt{23} \\ \frac{7}{1817} \sqrt{158} \sqrt{23} \\ -\frac{1}{1817} \sqrt{158} \sqrt{23} \end{pmatrix} = \frac{1}{23} \sqrt{158} \sqrt{23}, a_3 e_2 = \begin{pmatrix} 3 \\ 1 \\ -1 \\ 2 \end{pmatrix} \begin{pmatrix} \frac{53}{3634} \sqrt{158} \sqrt{23} \\ \frac{25}{3634} \sqrt{158} \sqrt{23} \\ \frac{7}{1817} \sqrt{158} \sqrt{23} \\ -\frac{1}{1817} \sqrt{158} \sqrt{23} \end{pmatrix} = \frac{83}{1817} \sqrt{158} \sqrt{23},$$

$$a_3 e_3 = \begin{pmatrix} 3 \\ 1 \\ -1 \\ 2 \end{pmatrix} \begin{pmatrix} \frac{32}{41949} \sqrt{531} \sqrt{79} \\ \frac{10}{13983} \sqrt{531} \sqrt{79} \\ -\frac{157}{41949} \sqrt{531} \sqrt{79} \\ \frac{124}{41949} \sqrt{531} \sqrt{79} \end{pmatrix} = \frac{3}{79} \sqrt{59} \sqrt{79}.$$

$$R = \begin{pmatrix} a_1 e_1 & a_2 e_1 & a_3 e_1 \\ 0 & a_2 e_2 & a_3 e_2 \\ 0 & 0 & a_3 e_3 \end{pmatrix} = \begin{pmatrix} \sqrt{23} & -\frac{7}{23} \sqrt{23} & \frac{4}{23} \sqrt{23} \\ 0 & \frac{1}{23} \sqrt{3634} & \frac{83}{1817} \sqrt{3634} \\ 0 & 0 & \frac{3}{79} \sqrt{4661} \end{pmatrix}.$$

$$\text{Finally } A = QR = \begin{pmatrix} \frac{1}{23} \sqrt{23} & \frac{53}{3634} \sqrt{3634} & \frac{32}{13983} \sqrt{4661} \\ -\frac{3}{23} \sqrt{23} & \frac{25}{3634} \sqrt{3634} & \frac{10}{4661} \sqrt{4661} \\ \frac{2}{23} \sqrt{23} & \frac{7}{1817} \sqrt{3634} & -\frac{157}{13983} \sqrt{4661} \\ \frac{3}{23} \sqrt{23} & -\frac{1}{1817} \sqrt{3634} & \frac{124}{13983} \sqrt{4661} \end{pmatrix} \begin{pmatrix} \sqrt{23} & -\frac{7}{23} \sqrt{23} & \frac{4}{23} \sqrt{23} \\ 0 & \frac{1}{23} \sqrt{3634} & \frac{83}{1817} \sqrt{3634} \\ 0 & 0 & \frac{3}{79} \sqrt{4661} \end{pmatrix}.$$